Hilbert Space

A Hilbert space is a complete normed linear space, $\mathbf{H}$, whose norm is defined by an inner product.

Complete means that all Cauchy sequences of vectors in $\mathbf{H}$ must converge in $\mathbf{H}$ with the norm from an inner product.

An *inner product* on a vector space $\mathbf{H}$ is a function

$$\langle \cdot , \cdot \rangle \colon \mathbf{H} \times \mathbf{H} \to \mathbb{C}$$

with the following properties ...

The Inner Product

... for all numbers α and vectors x, y and z

- $\langle x + y , z \rangle = \langle x , z \rangle + \langle y , z \rangle$ (additive in the first slot)
- $\langle \alpha x , z \rangle = \alpha \langle x , z \rangle$ (homogeneous in the first slot)
- $\langle x , y \rangle = \overline{\langle y , x \rangle}$ (conjugate symmetry)
- $\langle x , x \rangle \geq 0$ (positivity)
- $\langle x , x \rangle = 0 \Longrightarrow x = 0.$ (non-degeneracy)

$\|x\| = \sqrt{\langle x , x \rangle}$ defines the corresponding norm on $\mathbf{H}$.

(It is a small exercise to show $\|\alpha x\| = |\alpha|\|x\|$ and the triangle inequality, $\|x + y\| \leq \|x\| + \|y\|$.)

Examples

$\mathbb{C}^n$	$\langle x , y \rangle = \sum_{k=1}^{n} x_k \overline{y}_k$	$\|x\|^2 = \sum_{k=1}^{n} \|x_k\|^2$
$\mathbf{L}^2[0, 2\pi]$	$\langle f , g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}\, dx$	$\|f\|^2 = \int_0^{2\pi} \|f(x)\|^2\, dx$
ℓ^2	$\langle \{\alpha_k\} , \{\beta_k\} \rangle = \sum_{k=1}^{\infty} \alpha_k \overline{\beta}_k$	$\|\{\alpha_k\}\|^2 = \sum_{k=1}^{\infty} \|\alpha_k\|^2$

First Problem Set

$s(t) = t, \quad p(t) = t^2, \quad b(\theta) = e^{i\theta},$

$\vec{U} = (1, 0, 1), \quad \vec{V} = (-1, 1, 1),$

$\omega = \{0.5^n\}, \quad \lambda = \{0.2^n\}.$

(i) Determine the following inner products:

$\langle s , p \rangle \qquad \langle b , b \rangle \qquad \langle \vec{U} , \vec{V} \rangle \qquad \langle \omega , \lambda \rangle$

(ii) Determine the norms of the seven vectors listed above.

(iii) If $\chi \colon [0, 2\pi] \to \mathbb{R}$ is given by $\chi(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

then $\|\chi\| = 0$.

The Cauchy-Bunyakowski-Schwarz Inequality

$|\langle x, y \rangle| \leq \|x\| \, \|y\|$ for all x and y in Hilbert Space $\mathbf{H}$.

(Proof: Can assume $\|y\| = 1$. Then expand $\|x - \langle x, y \rangle y\|^2 \geq 0$ using properties of the inner product.)

The Inner Product is Continuous

Suppose $\mathbf{H}$ is a Hilbert Space.

The product space $\mathbf{H} \times \mathbf{H}$ is itself a vector space in a natural way, and a Hilbert Space with inner product $\langle \cdot, \cdot \rangle_{prod}$ given by

$$\langle (w, x), (y, z) \rangle_{prod} = \langle w, y \rangle + \langle x, z \rangle.$$

Convergence of a sequence (x_k, y_k) of vector pairs to a pair (x, y) means that $x_k \to x$ in $\mathbf{H}$ and also $y_k \to y$ in $\mathbf{H}$.

The inner product is a continuous function on this product space with the norm from this inner product, given by

$$\|(w, x)\|_{prod}^2 = \| w \|^2 + \|x\|^2.$$

(Proof: Use CBS+algebra+properties of the inner product.)

The Parallelogram Law

Suppose $\mathbf{H}$ is a Hilbert Space. The norm satisfies the *parallelegram law*,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(Proof: expand the inner products and compare.)

A Banach Space is a Hilbert Space if and only if the parallelegram law holds for its norm.

(Proof: $\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2 + i \, \|x + iy\|^2 - i \, \|x - iy\|^2$.)

Problem Set 2

Fix a vector x_0 in $\mathbf{H}$.

Define $f : \mathbf{H} \to \mathbb{C}$ by $f(x) = \langle x, x_0 \rangle$.

- Show that f is a linear functional.
- Show that $|f(x)| \leq \|x_0\|$ whenever $\|x\| \leq 1$.
- Show that f is continuous.

ORTHOGONALITY

Vectors x and y in Hilbert Space $\mathbf{H}$ are called *orthogonal* (to each other) if $\langle\, x\,,\, y\,\rangle = 0$.

We indicate this situation by $x \perp y$.

When $M \subset \mathbf{H}$ and $x \in \mathbf{H}$ we write $x \perp M$ when $x \perp y$ for every $y \in M$.

Finally, $M^{\perp} = \{\, x \in \mathbf{H} \mid x \perp M\,\}$.

$M^{\perp}$ is called *the orthogonal complement of* M.

EXAMPLES

In $\mathbb{C}^3$: $\vec{i}+\vec{j} \perp \vec{i}-\vec{j}$.

Also, $(a,b,c) \perp \{\,(t,u,v) \in \mathbb{C}^3 \mid at+bu+cv = 0\,\}$.

In $\mathbf{L}^2[0,2\pi]$: $\cos(\theta) \perp \sin(\theta)$ and $e^{i\theta} \perp e^{2i\theta}$

$\mathbf{1} \perp \{f \in \mathbf{L}^2[0,2\pi] \mid f(2\pi - x) = -f(x) \forall x \in [0,2\pi]\,\}$.

In ℓ^2 : Define for all $n \geq 1$

$\alpha_{2n} = 0 = \beta_{2n-1}$ and $\alpha_{2n-1} = \dfrac{1}{n} = \beta_{2n}$.

Then $\{\alpha_n\} \perp \{\beta_n\}$.

Also, if we define $\{b_n\}$ by $b_1 = b_2 = 1$ and $b_k = 0\ \forall k > 2$ then $\{b_n\} \perp \{\,\{a_n\} \in \ell^2 \mid a_1 = -a_2\,\}$.

THE PYTHAGOREAN THEOREM

$x \perp y$ when and only when

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

ORTHOGONAL SUBSPACES

We suppose $\mathbf{H}$ to be a nontrivial Hilbert Space (that is, it has dimension 1, at least) and M, M_1 and M_2 are nonempty subsets of $\mathbf{H}$.

$\{0\}^{\perp} = \mathbf{H}$ and $\mathbf{H}^{\perp} = \{0\}$.

$M \cap M^{\perp} = \{0\}$ or $\varnothing$.

$M_1 \subset M_2 \Longrightarrow M_2^{\perp} \subset M_1^{\perp}$.

$M^{\perp}$ is a *closed subspace* of $\mathbf{H}$.

If M is a *closed proper subspace* of $\mathbf{H}$ (that is, it is closed and not all of $\mathbf{H}$) then $M^{\perp}$ contains more than the zero vector.

If M and N are *closed subspaces* of $\mathbf{H}$ then so is $M+N$.

If M is a *closed subspace* of $\mathbf{H}$ then $\mathbf{H} = M \oplus M^{\perp}$.

The Riesz Representation Theorem

If f is a continuous linear functional on Hilbert Space $\mathbf{H}$ then there is a unique member y of $\mathbf{H}$ for which

$$f(x) = \langle x , y \rangle \text{ for all } x \in \mathbf{H}.$$

We will discuss the existence of orthonormal bases and Fourier expansions below, but an example of Riesz Representation in action is given by …

Riesz Rep in Action

Let $\{ \mathbf{e}_n \}$ be an orthonormal basis for $\mathbf{H}$. This implies that every $x \in \mathbf{H}$ can be written in a unique way as

$$x = \sum \alpha_n(x)\mathbf{e}_n$$

where the convergence of the series as a limit of partial sums is absolute in norm, in direct analogy to ordinary series convergence.

Define linear and bounded $S\colon \mathbf{H} \to \mathbb{C}$ and $T\colon \mathbf{H} \to \mathbb{C}$ by

$$S(x) = \alpha_1(x) \quad \text{and} \quad T(x) = \alpha_1(x) + \alpha_2(x) + \alpha_3(x).$$

The vectors that represent S and T, as guaranteed by the Riesz Representation Theorem, are $\mathbf{e}_1$ and $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$:

$$S(x) = \langle x, \mathbf{e}_1 \rangle \quad \text{and} \quad T(x) = \langle x, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \rangle.$$

Definition of Orthonormal Basis

Definition: A subset $B = \{\mathbf{e}_k\}$ of Hilbert Space $\mathbf{H}$ is called a **Hilbert basis (or orthonormal basis)** if it is both *complete* and *orthonormal*.

ortho means that the members of B are mutually orthogonal:

$$\mathbf{e}_i \perp \mathbf{e}_j \text{ when } i \neq j.$$

normal means that $\|\mathbf{e}_i\| = 1 \quad \forall i$.

complete means* that B is maximal: B is contained in no larger orthonormal subset of $\mathbf{H}$.

(* Note this second usage of the word *complete*, this time applied to a basis rather than a metric space.)

Facts about Bases

Zorn's Lemma implies that every Hilbert space contains an ONB.

For $x \in \mathbf{H}$ the set $\{ \mathbf{e}_k \mid \langle x, \mathbf{e}_k \rangle \neq 0 \}$ is countable.

For every $x \in \mathbf{H}$, $x = \sum \langle x, \mathbf{e}_k \rangle \mathbf{e}_k$. This is called a *Fourier expansion* of x in basis B, and the numbers $\langle x, \mathbf{e}_k \rangle$ are called *Fourier coefficients.*

The Gram-Schmidt orthogonalization process works in Hilbert space.

Examples of Bases

$\left\{ \mathbf{e}_n = \frac{e^{int}}{\sqrt{2\pi}} \mid n = 0, \pm 1, \pm 2, \dots \right\}$ is an ONB for $\mathbf{L}^2[0, 2\pi]$.

Completeness is hard and deep and equivalent to the topological completeness of $\mathbf{L}^2[0, 2\pi]$: the Riesz-Fisher theorem.

The Legendre polynomials = Gram-Schmidt applied to $\{ x^n \mid n = 0, 1, 2, \dots \}$ constitute an ONB for $\mathbf{L}^2[0, 2\pi]$.

Sequences

$$(1, 0, 0, 0, \dots), \ (0, 1, 0, 0, \dots), \ (0, 0, 1, 0, \dots), \ \dots$$

form an ONB for ℓ^2.