Spectral Theory Introduction

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Adjoint of a Bounded Operator

Suppose $\mathbf{H}$ is a Hilbert Space and $T \in B(\mathbf{H})$.

(That is, $T\colon \mathbf{H} \to \mathbf{H}$ is linear and bounded.)

The Adjoint of T is an operator T^* defined by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } x, y \in \mathbf{H}.$$

- Think *conjugate transpose*
- $\langle T^*x, y\rangle = \langle x, Ty\rangle$
- $A^* + B^* = (A+B)^*$
- $(AB)^* = B^*A^*$
- $T^{**} = T$

Self-Adjoint

T is called self-adjoint or Hermitian if $T^* = T$. The word Hermitian is universal in the finite dimensional setting: i.e. when T is a matrix.

Examples

- $E = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ for real a, b and c.
- $M(f(x)) = xf(x)$
- $A = TT^*$
- $Df = i\,\frac{df}{dx}$

A Proof Part One

Lemma

If $\langle Tx, x\rangle = 0$ for all x then $T = 0$.

Proof.

Suppose $\langle Tx, x\rangle = 0$ for all x. Then $\forall \alpha, \beta \in \mathbb{C}$ and $\forall x, y$

$$\begin{aligned} 0 &= \langle T(\alpha x + \beta y), \alpha x + \beta y\rangle \\ &= \langle T(\alpha x), \alpha x\rangle + \langle T(\alpha x), \beta y\rangle + \langle T(\beta y), \alpha x\rangle + \langle T(\beta y), \beta y\rangle \\ &= \langle T(\alpha x), \beta y\rangle + \langle T(\beta y), \alpha x\rangle = \alpha\overline{\beta}\langle Tx, y\rangle + \overline{\alpha}\beta\langle Ty, x\rangle. \end{aligned}$$

$\alpha = \beta = 1$ yields $\langle Tx, y\rangle + \langle Ty, x\rangle = 0$.
$\alpha = i$ and $\beta = 1$ yields $\langle Tx, y\rangle - \langle Ty, x\rangle = 0$.

Adding these two yields $\langle Tx, y\rangle = 0$ and the result follows. □

A Proof Part Two

Theorem

T is self-adjoint if and only if $\langle Tx, x\rangle$ is real for all $x \in \mathbf{H}$.

Proof.

If T is self-adjoint then $\overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle x, T^*x\rangle = \langle Tx, x\rangle$ so $\langle Tx, x\rangle$ is real for all x.

Conversely, if $\langle Tx, x\rangle$ is always real then

$$\langle T^*x, x\rangle = \langle x, Tx\rangle = \langle Tx, x\rangle.$$

This implies $\langle (T - T^*)x, x\rangle = 0\ \forall x$.
So $T = T^*$, by the preceding lemma. □

Projections

If $P^2 = P$ and $P^* = P$ then P is called an (orthogonal) projection.

P is said to be idempotent by virtue of the first property.

Examples

- $P_1 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$ and $P_2 = \begin{pmatrix} a & \pm\sqrt{a(1-a)} \\ \pm\sqrt{a(1-a)} & a \end{pmatrix}$.
- If M is a closed subspace of $\mathbf{H}$ and $x \in M$ and $x^\perp \in M^\perp$ define $P_3(x + x^\perp) = x$.
- If P is a projection and M is the range of P and N is the nullspace of P then $N = M^\perp$, $P(x) = x$ for all $x \in M$ and $I - P$ is the projection on $M^\perp$.
- If P_1 and P_2 are projections onto M_1 and M_2 respectively and $M_1 \perp M_2$ then $P_1P_2 = P_2P_1 = 0$ and $P_1 + P_2$ is the projection onto $M_1 \oplus M_2$.

Eigenthings

A nonzero vector x for which $Tx = \lambda x$ for some scalar λ is called an eigenvector for T and λ is called an eigenvalue for T.

The set $M_\lambda = \{\, x \in \mathbf{H} \mid (T - \lambda I)x = 0 \,\}$ is called the eigenspace for T and λ.

- The eigenvalues of $T = \begin{pmatrix} 1 & 0 & 0 \\ u & \pi & 0 \\ v & w & i \end{pmatrix}$ are $1, \pi$ and i.
  What are the eigenspaces?
- $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ viewed as a rotation in $\mathbb{R}^2$ has no real eigenvalues. But viewed as a operator on $\mathbb{C}^2$ it has eigenvalues $\pm i$ and eigenspaces M_i and M_{-i} spanned by $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ i \end{pmatrix}$, respectively.

More Eigenthings

- 0 and 1 are the only eigenvalues of projections.
- The forward shift S on ℓ^2 given by

$$S(z_0, z_1, z_2, \dots) = (0, z_0, z_1, z_2, \dots)$$

  has no eigenvalues.
- What are the eigenvalues and eigenvectors of the derivative operator? What about the *second* derivative operator?
- If λ is an eigenvalue of T then $\overline{\lambda}$ is an eigenvalue of T^*.

Still *More* Eigenthings

- Two other ways of saying that λ is an eigenvalue of T:
  1) $T - \lambda I$ is singular (*not* invertible)
  2) $\det(T - \lambda I) = 0$ *when* **H** *is finite dimensional.*
- Since $\det(T - \lambda I) = 0$ is an nth degree polynomial when $dim(\mathbf{H}) = n$ the second condition above and the Fundamental Theorem of Algebra guarantees that **all finite dimensional operators have lots of eigenvalues.**

Temporary Giant Assumptions

- $dim(\mathbf{H}) = n$.
- T has distinct eigenvalues $\lambda_1, \dots, \lambda_k$ with eigenspaces $M_1, \dots, M_k$.
- $M_i \perp M_j$ when $i \neq j$.
- $\mathbf{H} = M_1 \oplus M_2 \oplus \cdots \oplus M_k$.
- With these assumptions, every x in **H** has a unique representation as

$$x = x_1 + \cdots + x_k \quad \text{where each } x_i \in M_i.$$

An Implication of These Assumptions

- For every $x \in \mathbf{H}$ we have

$$\begin{aligned} Tx &= T(x_1 + \cdots + x_k) \\ &= T(x_1) + \cdots + T(x_k) = \lambda_1 x_1 + \cdots + \lambda_k x_k \\ &= \lambda_1 P_1(x) + \cdots + \lambda_k P_k(x) = (\lambda_1 P_1 + \cdots + \lambda_k P_k)(x) \end{aligned}$$

  where the P_i are the projections onto M_i.

  (That is, $P_i(x) = x_i$ and $P_i^2 = P_i$ and $P_i P_j = 0$ when $i \neq j$.)
- So $T = \lambda_1 P_1 + \cdots + \lambda_k P_k$.

  This is essentially the spectral theorem given our conditions.

Further Implications

- $$T^* = \overline{\lambda_1}\, P_1 + \cdots + \overline{\lambda_k}\, P_k.$$
- $$T\, T^* = |\lambda_1|^2\, P_1 + \cdots + |\lambda_k|^2\, P_k = T^*\, T.$$
- It can be shown that in finite dimensional spaces our special conditions hold exactly when $T^*\, T = T\, T^*$. Operators that commute with their adjoints are called normal operators.
- The finite dimensional spectral theorem implies that normal operators are diagonalizable.
- In particular self-adjoint operators on finite dimensional spaces are normal and hence diagonalizable.

Observation

- Define a new collection of orthogonal projections
$$E_0 = 0 \quad E_1 = P_1 \quad E_2 = P_1 + P_2 \quad E_3 = P_1 + P_2 + P_3$$
and carry on this way to $E_k = P_1 + \cdots + P_k$.
- $$\begin{aligned} T &= \lambda_1\, P_1 + \cdots + \lambda_k\, P_k \\ &= \lambda_1\,(E_1 - E_0) + \lambda_2\,(E_2 - E_1) \cdots + \lambda_k\,(E_k - E_{k-1}) \\ &= \sum_{i=1}^{k} \lambda_i\, \Delta\, E_i \end{aligned}$$
which looks a LOT like a generalized Riemann sum for
$$T = \int \lambda\, dE.$$

Preview

It is this form of the spectral theorem
$$T = \int \lambda\, dE$$
that applies to self-adjoint and normal operators on infinite dimensional Hilbert spaces.

The set of eigenvalues is replaced by the spectrum
$$\sigma(T) = \{\, \lambda \mid T - \lambda\, I \text{ is singular} \,\}$$
and ...lots of work. Among other things we will see that the spectrum is a compact set in the complex plane.

The Spectrum

The **spectrum** of a continuous operator $T\colon \mathbf{H} \to \mathbf{H}$ is the set of complex numbers
$$\sigma(T) = \{\, \lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible.} \,\}.$$

Recall that T will have an inverse function exactly when it is both one-to-one and onto.

These two properties are equivalent to each other in the finite dimensional setting, but not in the infinite dimensional case.
Remark:

The **Open Mapping Theorem** tells us that any continuous mapping from one Banach space to another that is onto the range space is an open mapping: it takes open sets in the domain space to open sets in the range space. In our situation, if T has an inverse function at all then that inverse function is continuous too.

Some Parts of the Spectrum

- There are two ways that an operator A can fail to have an inverse. There might be unequal x and y with $A(x) = A(y)$. Or the range $R(A)$ of A might not cover $\mathbf{H}$.
- **The point spectrum $\sigma_{\mathbf{p}}(\mathbf{T})$:** $\{ \lambda \mid T - \lambda I$ is not one-to-one. $\}$
- **The continuous spectrum $\sigma_{\mathbf{c}}(\mathbf{T})$:**
  $\{ \lambda \mid T - \lambda I$ is one-to-one and $R(T - \lambda I)$ is proper dense in $\mathbf{H}$. $\}$
- **The residual spectrum $\sigma_{\mathbf{r}}(\mathbf{T})$:**
  $\{ \lambda \mid T - \lambda I$ is one-to-one and $R(T - \lambda I)$ is not dense in $\mathbf{H}$. $\}$
- **The approximate point spectrum $\sigma_{\mathbf{ap}}(\mathbf{T})$:**
  $\{ \lambda \mid T - \lambda I$ is not bounded below. $\}$

  This last condition means that there is a sequence of unit vectors (x_n) for which $(T - \lambda I)x_n \to 0$.

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Some Facts about the Spectrum

- $\sigma(T)$ is a compact subset of the disk $\{ \lambda \in \mathbb{C} \mid \lambda \leq \|T\| \}$.
- $\sigma(T^*) = \overline{\sigma(T)} = \{ \lambda \in \mathbb{C} \mid \overline{\lambda} \in \sigma(T). \}$.
- If p is a polynomial then $\sigma(p(T)) = \{ p(\lambda) \in \mathbb{C} \mid \overline{\lambda} \in \sigma(T). \}$.
- $\sigma(T^{-1}) = \sigma(T)^{-1} = \{ \frac{1}{\lambda} \in \mathbb{C} \mid \overline{\lambda} \in \sigma(T). \}$.
- The boundary of $\sigma(T)$ is contained in the approximate point spectrum.
- If T is self-adjoint then $\sigma(T)$ is a subset of the real numbers and $\|T\|$ is the least upper bound of $\{ |\lambda| \mid \lambda \in \sigma(T) \}$. This least upper bound is called the **spectral radius r(T)** of T. (Note: $r(T) = \|T\|$ if T is any normal operator.)
- If T is self-adjoint then $\sigma(T)$ coincides with the approximate point spectrum. (Note: This is also true if T is any normal operator.)

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Examples involving the Spectrum

- If the dimension of $\mathbf{H}$ is finite then $\sigma(T) = \sigma_p(T)$. The spectrum is just the point spectrum.
- If T is a diagonal operator with square-summable diagonal entries $a_1, a_2, \ldots$ then the spectrum is the closure of this set of numbers, which are eigenvalues. If there is a limit point it must be 0 and if it is not an eigenvalue it is in the continuous spectrum.
- If S is a forward shift operator on ℓ_2 then the point spectrum is empty and $\sigma(S)$ is the closed unit disk and $\sigma_{ap}(S)$ is the closed unit circle.
- If $M(f) = xf$ for $f \in L^2[0,1]$ then M is self-adjoint and $\|M\| = \sup[0,1] = r(M) = 1$. The point spectrum of M is empty and $\sigma(M) = \sigma_{ap}(M) = [0,1]$.