EPR and the Bell and CSHS Inequalities

A Short Review

(Mike Ulrey March 2015)

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."
" (A. Einstein)

1. Quantum Mechanics Related to Pi Day (3/14/15)

Many persons have memorized large numbers of digits of π, a practice called piphilology.[137] One common technique is to memorize a story or poem in which the word lengths represent the digits of π: The first word has three letters, the second word has one, the third has four, the fourth has one, the fifth has five, and so on. An early example of a memorization aid, originally devised by English scientist James Jeans, is "How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics."[137] When a poem is used, it is sometimes referred to as a piem. Poems for memorizing π have been composed in several languages in addition to English.

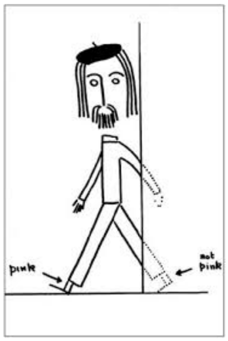
2. The Goals

Review EPR and the Bell inequalities

- **The common perception and Bertelman's socks**
- **The more nuanced version involving correlation probabilities**
- **Two proofs: Bell's original (Bell 1964) (simplified!) and the more recent and elegant CSHS version (Clauser, Horne, Shimony, and Holt 1969)**
- **If there is time, explore whether or not Bell's Theorem and the Bell Test Experiments (e.g., Aspect et al) are the final word on the EPR thought experiment. For example, how are the terms "local" and "hidden" expressed in the math? Are these concepts too tied up with classical probability, and would our conclusions change if we express them using quantum probability?**

Alice and Bob entanglement.

3. Bertelman's socks



Professor Bertelman (Hobbs?) ALWAYS wears socks of different color: pink and green.
When you see the pink sock, you know that the other sock (hidden from view) must be green.

Similarly, when two particles are prepared in an entangled state where one is spin-up and one is spin-down, why is it surprising that when one spin is measured you immediately know the other's spin orientation?

Bell's answer to the EPR thought experiment is more complicated than this. Why would that be necessary if the simple argument explains what's going on?

4. Bell's Original Paper (1964)

John Stewart Bell Irish physicist. On leave from CERN in 1964, spending time at Stanford, University of Wisconsin, and Brandeis, wrote "On the Einstein-Podolsky-Rosen Paradox".

Bell derived an inequality that any "local hidden variable theory" must satisfy, he claims. He then showed that this inequality is inconsistent with the predictions of QM. He thus turned the EPR argument on its head:

- **The EPR argument: "Entanglement (QM) + Locality ⟹ Hidden variables"**
- **Bell's argument: "Hidden variables + Locality ⟹ Not QM"**
- **Thus EPR is wrong.**
- **The proof is by contradiction: Bell derives a simple probabilistic inequality based on the hidden variable and locality assumptions. This inequality is contradicted by both QM theoretical predictions and several experiments.**

5. A quick review of EPR

The material on this slide is taken directly from Wikipedia <http://en.wikipedia.org/wiki/EPR_paradox> with slight edits. References are given at the bottom of the slide.

“The EPR paradox yields a dichotomy that physical reality as described by quantum mechanics is incomplete.

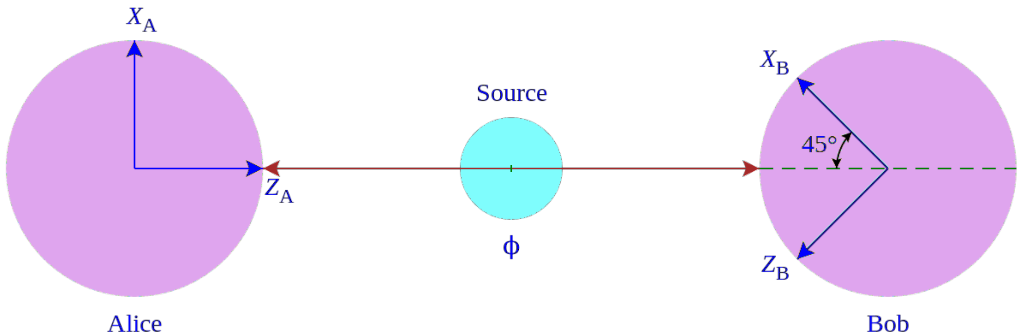
It is an early and influential critique levelled against the Copenhagen interpretation of quantum mechanics. Albert Einstein and his colleagues Boris Podolsky and Nathan Rosen (known collectively as EPR) designed a thought experiment [1] which revealed that the accepted formulation of quantum mechanics had a consequence which had not previously been noticed, but which looked unreasonable at the time. The scenario described involved the phenomenon that is now known as quantum entanglement.

According to quantum mechanics, under some conditions, a pair of quantum systems may be described by a single wave function, which encodes the probabilities of the outcomes of experiments that may be performed on the two systems, whether jointly or individually. At the time the EPR article was written, it was known from experiments that the outcome of an experiment sometimes cannot be uniquely predicted. An example of such indeterminacy can be seen when a beam of light is incident on a half-silvered mirror. One half of the beam will reflect, and the other will pass. If the intensity of the beam is reduced until only one photon is in transit at any time, whether that photon will reflect or transmit cannot be predicted quantum mechanically.

The routine explanation of this effect was, at that time, provided by Heisenberg's uncertainty principle. Physical quantities come in pairs called conjugate quantities. Examples of such conjugate pairs are position and momentum of a particle and components of spin measured around different axes. When one quantity was measured, and became determined, the conjugated quantity became indeterminate. Heisenberg explained this as a disturbance caused by measurement.

The EPR paper, written in 1935, was intended to illustrate that this explanation is inadequate. It considered two entangled particles, referred to as A and B, and pointed out that measuring a quantity of a particle A will cause the conjugated quantity of particle B to become undetermined, even if there was no contact, no classical disturbance. The basic idea was that the quantum states of two particles in a system cannot always be decom-

posed from the joint state of the two. An example (in bra–ket notation) is:
$|\Phi^{+}\rangle = (1/(\sqrt{2}))(|0, 0\rangle + |1, 1\rangle)$.



Heisenberg's principle was an attempt to provide a classical explanation of a quantum effect sometimes called non-locality. According to EPR there were two possible explanations. Either there was some interaction between the particles, even though they were separated, or the information about the outcome of all possible measurements was already present in both particles.

The EPR authors preferred the second explanation according to which that information was encoded in some 'hidden parameters'. The first explanation, that an effect propagated instantly, across a distance, is in conflict with the theory of relativity. They then concluded that quantum mechanics was incomplete since, in its formalism, there was no room for such hidden parameters.

Violations of the conclusions of Bell's theorem are generally understood to have demonstrated that the hypotheses of Bell's theorem, also assumed by Einstein, Podolsky and Rosen, do not apply in our world.[2] Most physicists who have examined the issue concur that experiments, such as those of Alain Aspect and his group, have confirmed that physical probabilities, as predicted by quantum theory, do exhibit the phenomena of Bell-inequality violations that are considered to invalidate EPR's preferred "local hidden-variables" type of explanation for the correlations to which EPR first drew attention.[3][4]"

1. Einstein, A; B Podolsky; N Rosen (1935-05-15). "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?". Physical Review 47 (10): 777–780.
2. Gaasbeek, Bram. "Demystifying the Delayed Choice Experiments", p. 1 (arXiv:1007.3977v1 [quant-ph] 22 Jul 2010)
3. Bell, John. On the Einstein–Poldolsky–Rosen paradox, Physics 1 3, 195-200, Nov. 1964
4. Aspect A (1999-03-18). "Bell's inequality test: more ideal than ever". Nature 398 (6724): 189–90.

6. Bell's response

The EPR paper and a rather opaque response by Neils Bohr (in a paper of the same title!) resulted in a kind of stalemate in the long-running debate between Einstein and Bohr. In fact it wasn't until almost 30 years later that Bell wrote his famous paper that produced a shocking conclusion concerning EPR's argument: *it is just plain wrong*.

The following description is lifted directly from "What is Bell's Theorem?" by Andrew Zimmerman Jones (http://physics.about.com/od/quantuminterpretations/f/bellstheorem.htm):

"One of the most curious elements of physics is the principle of quantum entanglement in quantum physics, where two seemingly independent particles appear to be connected to each other in a strange way. This behavior - which was famously debated by Albert Einstein and Niels Bohr - was called "spooky action at a distance" by Einstein.

However, physicist John Stewart Bell developed a way of determining whether this "action at a distance" (or non-local behavior, in more physics-like jargon) actually takes place. What was Bell's Theorem? Answer: Bell's Theorem was devised by Irish physicist John Stewart Bell (1928-1990) as a means of testing whether or not particles connected through quantum entanglement communicate information faster than the speed of light. Specifically, the theorem says that no theory of local hidden variables can account for all of the predictions of quantum mechanics. Bell proves this theorem through the creation of Bell inequalities, which are shown by experiment to be violated in quantum physics systems, thus proving that some idea at the heart of local hidden variables theories has to be false. The property which usually takes the fall is locality - the idea that no physical effects move faster than the speed of light.

John Stewart Bell originally proposed the idea for Bell's Theorem in his 1964 paper "On the Einstein Podolsky Rosen paradox." In his analysis, he derived formulas called the Bell inequalities, which are probabilistic statements about how often the spin of particle A and particle B should correlate with each other if normal probability (as opposed to quantum entanglement) were working. These Bell inequalities are violated by quantum physics experiments, which means that one of his basic assumptions had to be false, and there were only two assumptions that fit the bill - either physical reality or locality was failing.
To understand what this means, go back to the experiment described above. You measure particle A's spin. There are two situations that could be the result - either particle B immediately has the opposite spin, or particle B is still in a superposition of states. If particle B is affected immediately by the measurement of particle A, then this means that the assumption of locality is violated. In other words, somehow a "message" got from particle A to

particle B instantaneously, even though they can be separated by a great distance. This would mean that quantum mechanics displays the property of non-locality. If this instantaneous "message" (i.e., non-locality) doesn't take place, then the only other option is that particle B is still in a superposition of states. The measurement of particle B's spin should therefore be completely independent of the measurement of particle A, and the Bell inequalities represent the percent of the time when the spins of A and B should be correlated in this situation. Experiments have overwhelmingly shown that the Bell inequalities are violated. The most common interpretation of this result is that the "message" between A and B is instantaneous. (The alternative would be to invalidate the physical reality of B's spin.) Therefore, quantum mechanics seems to display non-locality. Note: This non-locality in quantum mechanics only relates to the specific information that is entangled between the two particles - the spin in the above example. The measurement of A cannot be used to instantly transmit any sort of other information to B at great distances, and no one observing B will be able to tell independently whether or not A was measured. Under the vast majority of interpretations by respected physicists, this does not allow communication faster than the speed of light.”

7. A note on a quantum physicist's use of the term "correlation"

The following is excerpted directly from Wikipedia: http://en.wikipedia.org/wiki/Bell%27s_theorem :

"The probability of the same result being obtained at the two locations varies, depending on the relative angles at which the two spin measurements are made, and is strictly between zero and one for all relative angles other than perfectly parallel alignments (0° or 180°). Bell's theorem is concerned with correlations defined in terms of averages taken over very many trials of the experiment. *The correlation of two binary variables is usually defined in quantum physics as the average of the product of the two outcomes of the pairs of measurements*. Note that this is different from the usual definition of correlation in statistics. *The quantum physicist's "correlation" is the statistician's "raw (uncentered, unnormalized) product moment"*. They are similar in that, with either definition, if the pairs of outcomes are always the same, the correlation is +1, no matter which same value each pair of outcomes have.[clarification needed] If the pairs of outcomes are always opposite, the correlation is -1. Finally, if the pairs of outcomes are perfectly balanced, being 50% of the times in accordance, and 50% of the times opposite, the correlation, being an average, is 0. The correlation is related in a simple way to the probability of equal outcomes, namely it is equal to twice this probability, minus one.

Measuring the spin of these entangled particles along anti-parallel directions—i.e., along the same axis but in opposite directions, the set of all results is perfectly correlated. On the other hand, if measurements are performed along parallel directions they always yield opposite results, and the set of measurements shows perfect anti-correlation. Finally, measurement at perpendicular directions has a 50% chance of matching, and the total set of measurements is uncorrelated. These basic cases are illustrated in the table below.

Anti-parallel	Pair 1	Pair 2	Pair 3	Pair 4	…	Pair n	
Alice, 0°	+	−	+	+	…	−	
Bob, 180°	+	−	+	+	…	−	
Correlation = (	+1	+1	+1	+1	…	+1	) / n = +1
(100% identical)							
Parallel	Pair 1	Pair 2	Pair 3	Pair 4	…	Pair n	
Alice, 0°	+	−	−	+	…	+	
Bob, 0° or 360°	−	+	+	−	…	−	

Correlation = (	-1	-1	-1	-1	…	-1	) / n = -1
(100% opposite)							
Orthogonal	Pair 1	Pair 2	Pair 3	Pair 4	…	Pair n	
Alice, 0°	+	−	+	−	…	−	
Bob, 90° or 270°		−	−	+	+	…	−
Correlation = (	−1	+1	+1	−1	…	+1	) / n = 0
(50% identical, 50% opposite)							

Figure 1

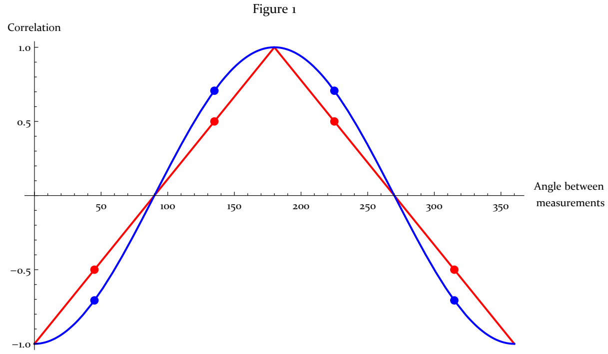


Figure 1: The best possible local realist imitation (red) for the quantum correlation of two spins in the singlet state (blue), insisting on perfect anti-correlation at zero degrees, perfect correlation at 180 degrees. Many other possibilities exist for the classical correlation subject to these side conditions, but all are characterized by sharp peaks (and valleys) at 0, 180, 360 degrees, and none has more extreme values (±0.5) at 45, 135, 225, 315 degrees. These values are marked by blue and red points in the graph, and are the values measured in a standard Bell-CHSH type experiment: QM allows $\pm\sqrt{\left(\frac{1}{2}\right)} = \pm 0.70$ ..., local realism predicts ±0.50 or less.

With the measurements oriented at intermediate angles between these basic cases, the existence of local hidden variables could agree with a linear dependence of the correlation in the angle but, according to Bell's inequality (see below), could not agree with the dependence predicted by quantum mechanical theory, namely, that the correlation is the negative cosine of the angle. Experimental results match the curve predicted by quantum mechanics.[1]

Over the years, Bell's theorem has undergone a wide variety of experimental tests. However, various common deficiencies in the testing of the theorem have been identified, including the detection loophole[7] and the communication loophole.[7] Over the years experiments have been gradually improved to better address these loopholes, but no experiment to date has simultaneously fully addressed all of them.[7] However, scientists generally expect that someone will conduct such an experiment in a few years, and it is expected to confirm yet again quantum predictions.[8] For example, Anthony Leggett has commented:

[While] no single existing experiment has simultaneously blocked all of the so-called loopholes, each one of those loopholes has been blocked in at least one experiment. Thus, to maintain a local hidden variable theory in the face of the existing experiments would appear to require belief in a very peculiar conspiracy of nature.[9]

To date, Bell's theorem is generally regarded as supported by a substantial body of evidence and there are few supporters of local hidden variables, though the theorem is continually subject of study, criticism, and refinement."

1. C.B. Parker (1994). McGraw-Hill Encyclopaedia of Physics (2nd ed.). McGraw-Hill. p. 542. ISBN 0-07-051400-3.
2. Mermin, David (April 1985). “Is the moon there when nobody looks? Reality and the quantum theory”. Physics Today: 38–47.
3. Stapp, Henry P. (1975). “Bell’s Theorem and World Process”. Nuovo Cimento 29B (2): 270. doi:10.1007/BF02728310. (Quote on p. 271)
4. Bell, John (1964). “On the Einstein Podolsky Rosen Paradox”. Physics 1 (3): 195–200.
5. The quotation is an adaptation from the edited transcript of the radio interview with John Bell of 1985. See The Ghost in the Atom: A Discussion of the Mysteries of Quantum Physics, by Paul C. W. Davies and Julian R. Brown, 1986/1993, pp. 45-46
6. Bohm, David (1951). Quantum Theory. Prentice–Hall.
7. Article on Bell’s Theorem by Abner Shimony in the Stanford Encyclopedia of Philosophy, (2004).

8. Derivation of the Bell inequality

Imagine Alice and Bob each receive one of two entangled particles in a total spin 0 state, as in the EPR thought experiment.

- **Alice can measure spin components A or B**
- **Bob can measure spin components C or D**
- **Thus there are four possible joint measurements (A, C), (A, D), (B, C), (B, D).**
- **Let $P(X = Y)$ denote the probability that the X and Y measurements give the same result, either both $\hbar/2$ or both $-\hbar/2$.**
- **By the hidden variables assumption, A, B, C, D all have values every time we do the experiment, even though we only find out some of the values.**
- **By the locality assumption, the *value* of Alice's measurement of A does not depend on whether Bob *measures* C or D, for example.**
- **Note that $(A = C) \wedge (B = D) \wedge (C = B) \Longrightarrow (A = D)$, so $P((A = C) \wedge (B = D) \wedge (C = B)) \leq P(A = D)$, and thus (by one of de Morgan's laws and simple probability), $P((A \neq C) \vee (B \neq D) \vee (C \neq B)) \geq P(A \neq D) = 1 - P(A = D)$**
- **Assume: $P(A = C) = P(B = D) = 0.85$ and $P(C = B) = 1$. Then**
  - $P(A = D) \geq 1 - P((A \neq C) \vee (B \neq D) \vee (C \neq B)) \geq 1 - P(A \neq C) - P(B \neq D) - P(C \neq B)) =$
    $1 - 0.15 - 0.15 - 0 = .70$
- **But QM predicts that if we create two spins in a total spin 0 state, then the probability of agreement between two spin measurements, where α is the angle between the measurement axes, is given by (Note: *We will derive this relationship in general on the next few slides*) :**

Table 1
Probability of agreement vs difference between measuring angles

α	0	45 °	90 °	135 °
Probability of agreement	0	0.15	0.5	0.85

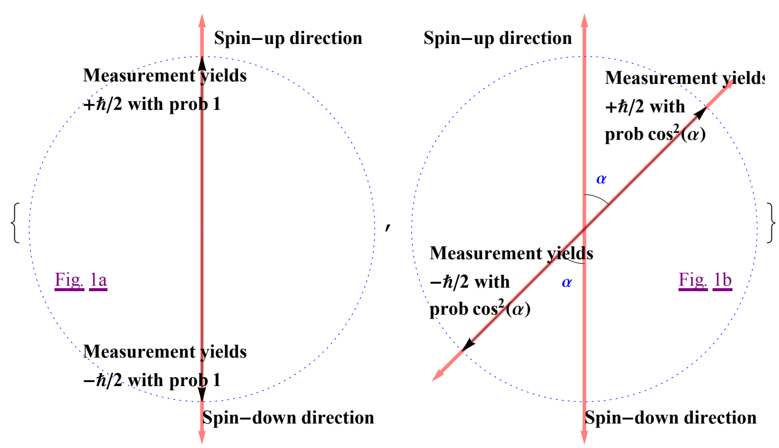
- **We choose 4 spin axes so that ∢AC=135°, ∢BC=180°, ∢BD=135° and ∢AD=90°**
- **Thus our assumptions are satisfied: $P(A = C) = P(B = D) = 0.85$ and $P(C = B) = 1$, so we get $P(A = D) \geq .70$**
- **But as seen in the table above for a 90° angle, QM predicts that $P(A = D) = .50 < .70$, hence the Bell inequality is violated!**

9. Derivation of the QM prediction

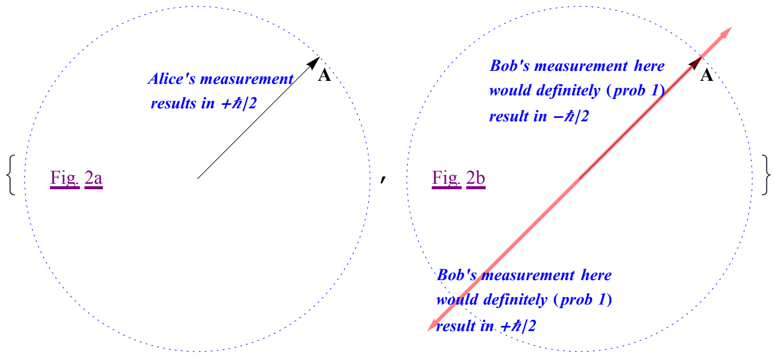
We use the following tenets of QM regarding measurement of a particle spin in the xz-plane for a single spin-1/2 system (see Appendices 1 and 2 for more details):

1. No matter the angle chosen for the measurement, the result is either $+\hbar/2$ or $-\hbar/2$.
2. If the angle between a particle's spin-up axis (meaning a measurement oriented with this direction will produce $+\hbar/2$ with probability 1), and the measurement axis makes an angle α with the spin-up axis, then the probability of getting $+\hbar/2$ is $\cos^2(\alpha/2)$ and the probability of getting $-\hbar/2$ is $\sin^2(\alpha/2)$. See Appendices 3, 4, and 5.
3. In the case of an entangled state of *two quibit* systems (two 2-dimensional Hilbert spaces), neither system has a definite state until the entangled state is measured. The state of the one system is then oriented in the direction of the measurement after the measurement is made, i.e., the indefinite state "collapses" to a definite observed state, namely spin-up. The state of the other system collapses to a spin-down state, i.e., would yield $+\hbar/2$ if measured 180° out of phase with the first measurement, but $-\hbar/2$ if measured in the same direction as the first measurement.

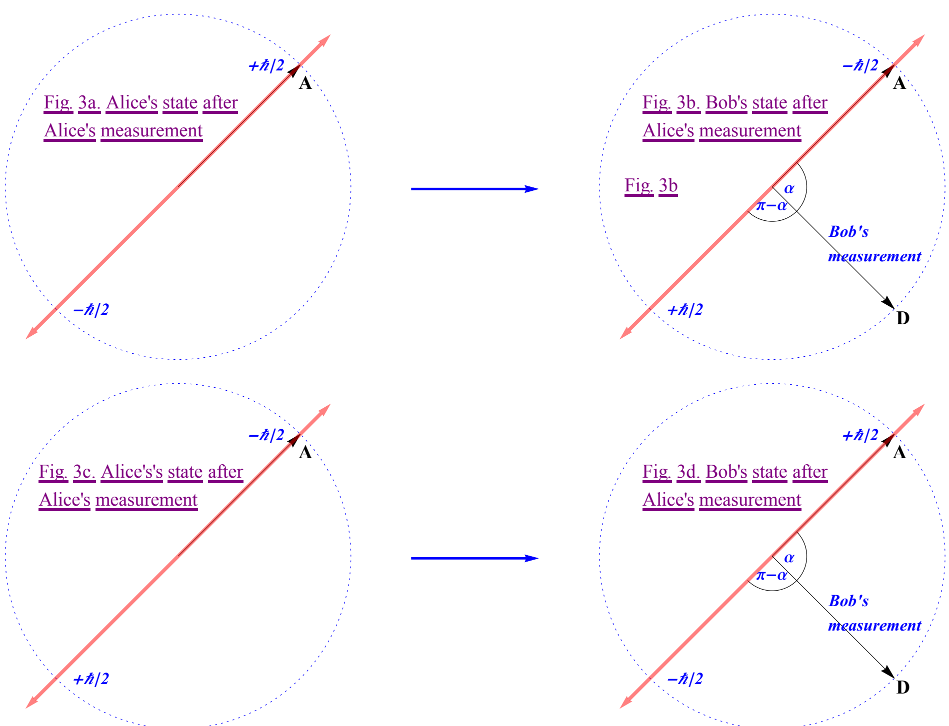
Consider a single (un-entangled) system, e.g., a particle with a definite (probability 1) spin-up (and spin-down) direction. See Figures 1a and 1b below. In Fig. 1a, the measurement has been made in the spin-up direction, so results in $+\hbar/2$ with probability 1 (and $-\hbar/2$ with probability 0). In Fig. 1b, if a measurement is made at an angle α from the spin-up direction, the result will be $+\hbar/2$ with probability $\cos^2(\alpha/2)$, and $-\hbar/2$ with probability $\sin^2(\alpha/2)$. Similarly, if a measurement is made at an angle α from the spin-down direction, the result will be $-\hbar/2$ with probability $\cos^2(\alpha/2)$, and $+\hbar/2$ with probability $\sin^2(\alpha/2)$.



For an entangled system, consider Figures 2a and 2b below. In Fig. 2a, Alice's measurement has been made at the orientation shown, and results in $+\hbar/2$. The system then "collapses" into the measured state, resulting in the spin-up and spin-down directions as shown in Fig. 2b. If Bob were to make a measurement at the same orientation as Alice, he would get $-\hbar/2$ with probability 1, and if he were to measure at an orientation 180° out of phase with Alice's measurement, he would get $+\hbar/2$ with probability 1 .



Consider the same situation, except Bob's measurement orientation makes an angle α with that previously made by Alice. In the first row below (Figures 3a and 3b) Alice's measurement A has resulted in $+\hbar/2$, and Bob gets $+\hbar/2$ with probability $\cos^2((\pi-\alpha)/2) = \sin^2(\alpha/2)$. In the second row below (Figures 3c and 3d) Alice's measurement A has resulted in $-\hbar/2$, and Bob gets $-\hbar/2$ also with probability $\cos^2((\pi-\alpha)/2) = \sin^2(\alpha/2)$.



Bob's measurement yields + with probability $\cos^2((\pi-\alpha)$

Bob's measurement yields – with probability $\cos^2((\pi-\alpha)$

Define

A_+ = measurement A yields spin up $(+\hbar/2)$

A_- = measurement A yields spin down ($-\hbar/2$)
D_+ = measurement D yields spin up (relative to the state after Alice's measurement)
D_- = measurement D yields spin down (relative to the state after Alice's measurement).

Then the event $A = D$ occurs iff both A_+ and D_+ occur or both A_- and D_- occur. Therefore

$P(A = D) = P(A_+)\, P(D_+ \mid A_+) + P(A_-)\, P(D_- \mid A_-)$,

where the second probabilities in each term are conditional on the outcome of the first, since the state that Bob measures against is the one that is produced after Alice's measurement (see QM rule #3 above). Then from the discussion before Figures 3a-3d above, we have that $P(D_+ \mid A_+) = P(D_- \mid A_-) = \mathrm{Sin}^2[\alpha/2]$, and therefore,

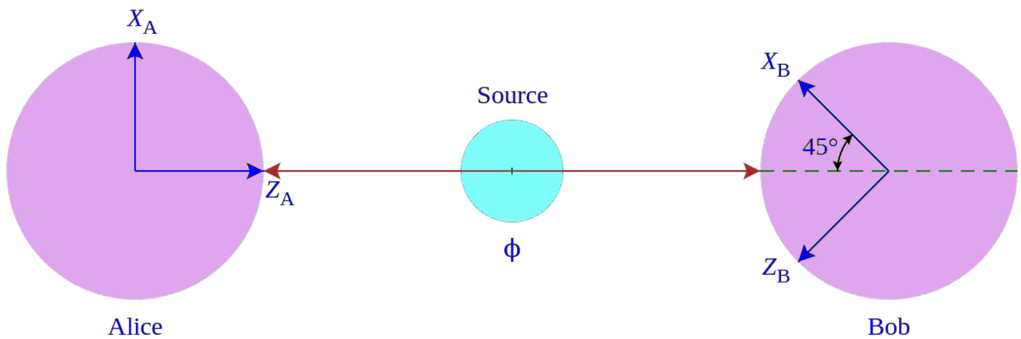
$P(A = D) = P(A_+)\, P(D_+ \mid A_+) + P(A_-)\, P(D_- \mid A_-) =$
$\quad P(A_+)\, \mathrm{Sin}^2(\alpha/2) + P(A_-)\, \mathrm{Sin}^2(\alpha/2) = (P(A_+) + P(A_-))\, \mathrm{Sin}^2(\alpha/2) = \mathrm{Sin}^2(\alpha/2)$
(notice that Alice's probabilities "drop out", i.e., it doesn't matter what they are)

If $\alpha = 90\,°$, then $P(A = D) = \mathrm{Sin}^2[45\,°] = 0.50$, as claimed.

10. Another demonstration of Bell’s result: CHSH

(Clauser, Horne, Shimony, and Holt 1969)

(CHSH slides based on Sec 6.6 of "Quantum Processes, Systems, and Information", Schumacher and Westmoreland)



- **A source produces pairs of physical systems and routes one of the pair to Alice and one to Bob, who are separated observers.**
- **Alice can measure observables A_1 and A_2 on her system and Bob can measure observables B_1 and B_2 on his.**
- **There are four possible joint measurements (A_1, B_1), (A_1, B_2), (A_2, B_1), and (A_2, B_2).**
- **They can only measure one pair at any given time, but can run the experiment as many times as they want.**
- **For simplicity, assume the observables can only take the values -1 and $+1$.**

11. The CHSH inequality

Furthermore, assume (more or less in line with the EPR point of view)

- **Hidden variables: The results of any measurement on any individual system are pre-determined. Any probabilities we may use only reflect our ignorance of the (hidden) definite values.**
- **Locality: Alice's choice of measurement does not affect the outcomes of Bob's measurements, and *vice versa*.**
  - **The statement "The value of B_1 is +1" means "If Bob were to measure B_1, then the result +1 would be obtained", regardless of whether Alice chooses to measure A_1 or A_2. (Note Alice's *outcome* may affect Bob's probabilities, but this does not allow Alice to send a message to Bob (also see the "no-communication" theorem)).**
  - **This is where the hidden variables and locality assumptions come in.**
    - **Hidden variables: All four measurements A_1, A_2, B_1, B_2 have definite values that are predetermined, that is, exist before any measurement is made.**
    - **Locality: Alice's choice of measurement does not affect Bob's outcome, and vice-versa.**
    - **As a result of these assumptions, we conclude that $B_1 + B_2$ can only take on values -2, 0, or +2 and $B_1 - B_2$ can take on values 0, ±2, or 0, respectively**
- **This allows us to form the observable $Q = A_1(B_1 - B_2) + A_2(B_1 + B_2)$ and make certain conclusions that follow below:**
- **Since A_i, $B_i = \pm 1$ for $i = 1, 2$, we have that $Q = \pm 2$**
- **Thus the expected value of the random variable Q is constrained by $-2 \leq \langle Q \rangle \leq 2$,**
- **Hence, by expanding the expression for Q and using the linearity of expectation, we get:**
- **$-2 \leq \langle A_1 B_1 \rangle - \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle + \langle A_2 B_2 \rangle \leq +2$, where $\langle A_i B_j \rangle$ denotes the expected value of the product of A_i and B_j.**
- **This is called a CHSH inequality, a specific example of a Bell-type inequality.**
- **Neither Alice nor Bob can determine the value of Q in any single experiment, but they can (jointly) measure any of the products $A_i B_j$.**
- **By repeating the experiment many times, statistical averages can be computed for each of the terms in the inequality, providing estimates of the expectations.**
- **This produces an experimentally testable statement based on the assumptions of hidden variables and locality.**

12. Evaluation of $Q = A_1(B_1 - B_2) + A_2(B_1 + B_2)$ for an entangled state

- **Let $|\Psi_-^{(AB)}\rangle = (1/\sqrt{2})(|0,1\rangle - |1,0\rangle)$ and $|\Phi_+^{(AB)}\rangle = (1/\sqrt{2})(|0,0\rangle + |1,1\rangle)$ (each is an entangled state)**
- **Alice and Bob will measure observables associated with operators of the form $W_\theta = \sin\theta\, X + \cos\theta\, Z$, where X and Z are the Pauli operators associated with the x and z directions (see Appendix 4). This is a measurement in the xz-plane.**
- **Alice measures W_θ and Bob measures W_{θ^*}. Then for the quantum state $|\Psi_-^{(AB)}\rangle = (1/\sqrt{2})(|0,1\rangle - |1,0\rangle)$, the expectation (see Appendix 1 and Appendix 7):**
  - **$\langle W_\theta W_{\theta^*}\rangle = -\sin\theta\sin\theta^* - \cos\theta\cos\theta^* = -\cos(\theta - \theta^*)$.**

Proof:

From Appendix 6 we have

$Z^{(A)} Z^{(B)} |\Psi_-^{(AB)}\rangle = -|\Psi_-^{(AB)}\rangle$

$X^{(A)} X^{(B)} |\Psi_-^{(AB)}\rangle = -|\Psi_-^{(AB)}\rangle$ **(Eq 1)**

$X^{(A)} Z^{(B)} |\Psi_-^{(AB)}\rangle = -|\Phi_+^{(AB)}\rangle$

$Z^{(A)} X^{(B)} |\Psi_-^{(AB)}\rangle = +|\Phi_+^{(AB)}\rangle$

From Appendix 7 we have

$\langle Z^{(A)} Z^{(B)}\rangle = -1$

$\langle X^{(A)} X^{(B)}\rangle = -1$ (Eq 2)

$\langle X^{(A)} Z^{(B)}\rangle = 0$

$\langle Z^{(A)} X^{(B)}\rangle = 0$

Therefore,

$$\langle W_\theta W_{\theta^*}\rangle = \langle(\sin\theta\, X^{(A)} + \cos\theta\, Z^{(A)})(\sin\theta^*\, X^{(B)} + \cos\theta^*\, Z^{(B)})\rangle =$$
$$\sin\theta\sin\theta^*\langle X^{(A)} X^{(B)}\rangle + \sin\theta\cos\theta^*\langle X^{(A)} Z^{(B)}\rangle + \cos\theta\sin\theta^*\langle Z^{(A)} X^{(B)}\rangle +$$
$$\cos\theta\sin\theta^*\langle Z^{(A)} Z^{(B)}\rangle = -\sin\theta\sin\theta^* - \cos\theta\sin\theta^* = -\cos(\theta - \theta^*)$$

- **The equation above results from the linearity of the expectation operator and the values for the paired expectations we derived in equations (2) above.**
- **Now choose $A_1 = W_0$, $B_1 = W_{\pi/4}$, $A_2 = W_{\pi/2}$, and $B_2 = W_{3\pi/4}$.**
- **With these choices $\langle A_1 B_1\rangle = -1/\sqrt{2}$, $\langle A_1 B_2\rangle = 1/\sqrt{2}$, $\langle A_2 B_1\rangle = -1/\sqrt{2}$, and $\langle A_2 B_2\rangle = -1/\sqrt{2}$, and so.... (drum roll please!)**
- **$Q = \langle A_1 B_1\rangle - \langle A_1 B_2\rangle + \langle A_2 B_1\rangle + \langle A_2 B_2\rangle = -2\sqrt{2}$, thus violating the CHSH (Bell) inequality! Therefore...**
- **Hidden variables + Locality $\Longrightarrow$ Not QM (since CHSH inequality has been shown experimentally NOT to hold in certain cases)**

13. J. S. Bell’s View on Hidden Variables

From “Introduction to the Hidden Variable Question” --
On the topic of the boundary between the classical and quantum domains, he writes:
“A possibility is that we find exactly where the boundary lies. More plausible to me is that we will find that there is no boundary. It is hard for me to envisage intelligible discourse about a world with no classical part -- no base of given events, be they only mental events in a single consciousness, to be correlated. On the other hand, it is easy to imagine the classical domain could be extended to cover the whole. The wave functions would prove to be a provisional or incomplete description of the quantum-mechanical part, of which an objective account would become possible. It is this possibility, of a homogeneous account of the world, which is for me the chief motivation of the study of the so-called “hidden variable” possibility.

14. Where do we go from here?

Possibilities include:

1. Dirac's solutions of the Schrodinger wave equation with a relativistic Hamiltonian. Why? Because this is the theory that led to a mathematical model of "spin" that predicts spin behavior very precisely. The four-vectors of the relativistic theory replace the scalars of the non-relativistic theory. These give a more specific meaning to the abstract concept of spin states that we have considered in this presentation. For example, we would be considering 4-vectors of the form

$$\begin{pmatrix} 0 \\ i\,b\,k\cos\left(\frac{k\pi x_1}{L}\right) \\ \sin\left(\frac{k\pi x_1}{L}\right) \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} i\,b\,n\cos\left(\frac{k\pi x_2}{L}\right) \\ 0 \\ 0 \\ \sin\left(\frac{k\pi x_2}{L}\right) \end{pmatrix}$$

and their various combinations. This would connect to Victor's material and the traditional wave function perspective of QM and related experiments.

2. Tensor products. Understanding the tensor product of Hilbert spaces, states, and operators is essential to a deep understanding of entanglement.

3. Stanley Gudder's claims for the power of generalized probability spaces to resolve some QM "paradoxes". Is this for real? Does it really provide a consistent perspective on hidden variables that resolves the EPR-Bell Theorem clash but still agrees with QM predictions and experiments? See sections 2.4 (p 54) and 6.3 (p 224) and 6.4 (p 228) of Gudder's book in the references.

15. References

Papers and books

"Bertelman's Socks and the Nature of Reality", J. S. Bell, Journal de Physique, Colloqie C2, supplement au no 3, Tome 42, mars 1981 https://cds.cern.ch/record/142461/files/198009299.pdf

"Quantum Processes, Systems, and Information", Benjamin Schumacher and Michael Westmoreland, Cambridge University Press, 2010

"Quantum Probability", Stanley Gudder; Academic Press, 1988

"Introduction to Quantum Mechanics", Chalmers Sherwin; Holt,rinehart and Winston, 1959

Online resources and YouTube videos

Various

<http://susanka.org/HSforQM/>
See BC QM website maintained by Larry Susunka. For example. Larry's lecture series on tensor products or the collection of J. S. Bell papers. Excellent writing in both!

EPR

https://www.youtube.com/watch?v=5HJK5tQIT4A
http://en.wikipedia.org/wiki/EPR_paradox

Spin

https://www.youtube.com/watch?v=v1_-LsQLwkA

Bell inequalities and test experiments

https://www.youtube.com/watch?v=z-s3q9wlLag

https://www.youtube.com/watch?v=7zfnvGXpy-g

https://www.youtube.com/watch?v=8UxYKN1q5sI

http://en.wikipedia.org/wiki/Bell_test_experiments

Bohmian mechanics or Pilot-Wave Theory

https://www.youtube.com/watch?v=rbRVnC92sMs

https://www.youtube.com/watch?v=Qz4CHI_W-TA

Appendix 1: Some notation and terminology

(Based on notation from "Quantum Processes, Systems, and Information", Schumacher and Westmoreland).

We talk mostly about finite-dimensional Hilbert spaces in the following, in fact 2-dimensional for the key example.

- **An *observable* is a basic measurement in which each outcome is associated with a numerical value.**
- **Suppose A_n is the nth outcome, associated with a basis element $|n\rangle$.**
- **The operator A associated with a measurement A is defined by $\mathrm{A}\,|n\rangle = A_n\,|n\rangle$ for all $|n\rangle$.**
- **The operator A can be written as $\mathrm{A} = \sum_n A_n\,|n\rangle\langle n|$, where**
  - **The *outer product* $|\alpha\rangle\langle\beta|$ is defined by $|\alpha\rangle\langle\beta|(|\psi\rangle) \doteq |\alpha\rangle\langle\beta|\psi\rangle = \langle\beta|\psi\rangle\,|\alpha\rangle$ for all states $|\psi\rangle \in \mathcal{H}$.**
  - **In a two-dimensional Hilbert space with (orthonormal) basis vectors $|0\rangle$ and $|1\rangle$, and $|\psi\rangle = c_0\,|0\rangle + c_1\,|1\rangle$, then $|0\rangle\langle 0|\,|\psi\rangle = c_0\,|0\rangle\langle 0|\,(|0\rangle + c_1\,|0\rangle\langle 0|\,|1\rangle = c_0\,\langle 0\,|0\rangle\,|0\rangle + c_1\,\langle 0\,|1\rangle\,|0\rangle = c_0\,|0\rangle$, i.e.,**
  - **$|0\rangle\langle 0|$ is the projection onto $\langle 0\,|$. Likewise, $|1\rangle\langle 1|$ is the projection onto $\langle 1\,|$.**
- **For an observable A, $\langle A\rangle \doteq \langle\psi\,|\mathrm{A}|\,\psi\rangle$ denotes the *expected value* (or *expectation*) of A (relative to the state ψ).**
- **If the state-space is finite-dimensional and $\mathcal{A}$ is the matrix representing A in some basis, and ψ is the (column) vector of components of $|\psi\rangle$ in the same basis, then $\langle A\rangle = \psi^\dagger\,\mathcal{A}\,\psi$.**
- **A *qubit* system is a term for the simplest generic kind of quantum system. It has two distinguishable states $|0\rangle$ and $|1\rangle$, say, which form an orthonormal basis, and all other states are superpositions $|\psi\rangle = \alpha_0\,|0\rangle + \alpha_1\,|1\rangle$.**
- **The tensor product of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is denoted $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and the elements of $\mathcal{H}$ are denoted by**
  - **$|a,\,b\rangle = |a\rangle \otimes |b\rangle$ ((tensor) *product* vectors)**
  - **$\otimes$ distributes over linear combinations: $|a\rangle \otimes (c_1\,|b_1\rangle + c_2\,|b_2\rangle) = c_1(|a\rangle \otimes |b_1\rangle) + c_2(|a\rangle \otimes |b_2\rangle)$.**
  - **$\mathcal{H}$ contains *only* product vectors or linear combinations of product vectors.**
  - **The inner product in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is defined by $\langle a_1,\,b_1\,|a_2,\,b_2\rangle = \langle a_1 \otimes b_1\,|a_2 \otimes b_2\rangle \doteq \langle a_1\,|a_2\rangle\,\langle b_1\,|b_2\rangle$, where of course the first inner product is in $\mathcal{H}_1$ and the second is in $\mathcal{H}_2$.**

Appendix 2: A brief review of particle spin math (1)

Notation and Fundamentals for spin-1/2 particles:

- **Electrons, protons, and neutrons are examples of spin-1/2 particles**
- **The result of a measurement of the spin angular momentum is either $+\hbar/2$ or $-\hbar/2$ *for any component of the spin.***
  - **This is an empirical observation, e.g., the Stern-Gerlach experiment.**
  - **The state corresponding to $+\hbar/2$ is called "spin up" and to $-\hbar/2$ is called "spin down"**

Probability amplitude vectors:

- **Imagine the Stern-Gerlach apparatus oriented so that particles go "up" along the positive z-axis or "down" along the negative z-axis.**
- **Then the probability amplitude vectors along the three axes are given by:**
  - $x_+ = (1/(\sqrt{2}))\begin{pmatrix}1\\1\end{pmatrix}$ **and** $x_- = (1/(\sqrt{2}))\begin{pmatrix}1\\-1\end{pmatrix}$
  - $y_+ = (1/(\sqrt{2}))\begin{pmatrix}1\\i\end{pmatrix}$ **and** $y_- = (1/(\sqrt{2}))\begin{pmatrix}1\\-i\end{pmatrix}$
  - $z_+ = \begin{pmatrix}1\\0\end{pmatrix}$ **and** $z_- = \begin{pmatrix}0\\1\end{pmatrix}$
- **Notice that ,** $z_- = (1/(\sqrt{2}))\,x_+ - (1/(\sqrt{2}))\,x_- = -(i/(\sqrt{2}))\,y_+ + (i/(\sqrt{2}))\,y_-$.
- **Thus, for example, if a particle is prepared so that it is *definitely* (probability 1) spin up or spin down along the x-axis but is measured along the z-axis (attained by appropriate orientation of the Stern-Gerlach apparatus, for example), then the probabilities for measuring spin up ($+\hbar/2$) or spin down ($-\hbar/2$) along the z-axis are each $(1/(\sqrt{2}))^2 = \frac{1}{2}$, since these are the squares of the coefficients of x_+ and x_-, and hence the probabilities of getting $+\hbar/2$ and $-\hbar/2$.**
- **Similarly, the probabilities are also $\frac{1}{2}$ and $\frac{1}{2}$ if the particle is prepared so that it's spin is *definitely* (probability 1) spin up or spin down along the y-axis, but measured along the z-axis.**

Appendix 3: A brief review of particle spin math (2)

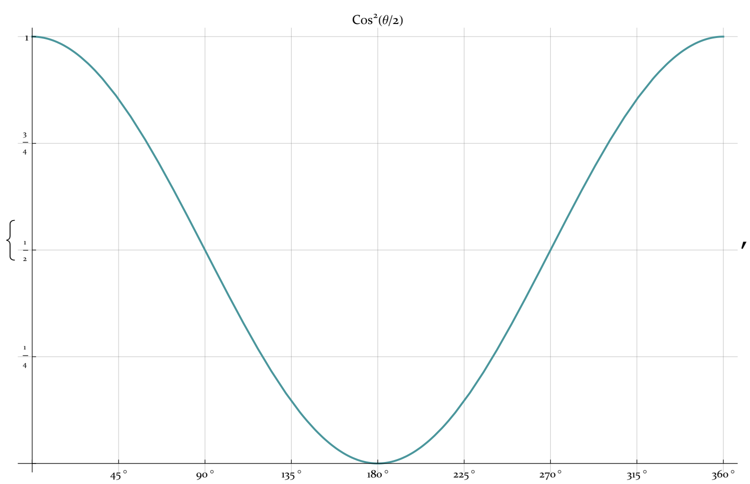
In general, consider a direction in the xz-plane that makes an angle θ with the z-axis. Suppose $\boldsymbol{\theta}_+ = \begin{pmatrix} \text{Cos}[\theta/2] \\ \text{Sin}(\theta/2] \end{pmatrix}$ and $\boldsymbol{\theta}_- = \begin{pmatrix} -\text{Sin}[\theta/2] \\ \text{Cos}(\theta/2] \end{pmatrix}$ are probability amplitude vectors where the spin component S_θ has *definite* (probability 1) values $+\hbar/2$ or $-\hbar/2$. *See Appendix 6 for more details and proofs.*

What is the probability that a measurement of S_z is $+\hbar/2$ (spin up)?

Since $\mathbf{z}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} \text{Cos}[\theta/2] \\ \text{Sin}[\theta/2] \end{pmatrix} + c_2 \begin{pmatrix} -\text{Sin}[\theta/2] \\ \text{Cos}(\theta/2] \end{pmatrix}$, we have $c_1 = \text{Cos}[\theta/2]$ and $c_2 = -\text{Sin}[\theta/2]$.

Hence probability of a z-measurement yielding $+\hbar/2$ is
$|c_1|^2 = \text{Cos}^2(\theta/2)$ (and probability of $-\hbar/2$ is $|c_2|^2 = \text{Sin}^2(\theta/2)$).

Consider the plot and table below which relates the probabilities of measuring $+\hbar/2$ in the z-direction given the particle is prepared in a state θ_+, for selected values of θ. This will be important for deriving a form of Bell's inequality.



Angle	0	45 °	90 °	135 °	180 °	225 °	270 °	315 °	360 °
Probability	1.0	0.85	0.50	0.15	0	0.15	0.50	0.85	1.0

Note that the cases of 0° and 180° correspond to the perfectly correlated and anti-correlated situations, respectively. A perfectly correlated situation would be one where the Stern-Gerlach apparatus is aligned perfectly with the way the particle was "prepared".

The anti-correlated situation is similar to Bertleman's socks. If one sock is known to be pink, the other must be non-pink (180 degrees out of phase).

Appendix 4: A Brief Review of Particle Spin Math (3)

- **The Pauli operators for a qubit system Q, and their matrix representations in the standard basis:**

  $\mathbb{X} = |0\rangle\langle 1| + |1\rangle\langle 0|$; $X = \text{matrix}(\mathbb{X}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  $\mathbb{Y} = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|$; $Y = \text{matrix}(\mathbb{Y}) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

  $\mathbb{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$; $Z = \text{matrix}(\mathbb{Z}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- **Let S_x, S_y, S_z denote the components of a spin vector S. The table below shows some notation related to these observables.**

Spin component observable	Associated operator	In terms of Pauli operator	In terms of own basis	In terms of 'standard' basis	In terms of z basis
S_x	$\mathbf{S_x}$	$\hbar/2\ \mathbb{X}$	$\hbar/2(|x_+\rangle\langle x_+| - |x_-\rangle\langle x_-|)$	$\hbar/2(|0\rangle\langle 1| + |1\rangle\langle 0|)$	$\hbar/2(|z_+\rangle\langle z_-| + |z_-\rangle\langle z_+|)$
S_y	$\mathbf{S_y}$	$\hbar/2\ \mathbb{Y}$	$\hbar/2(|y_+\rangle\langle y_+| - |y_-\rangle\langle y_-|)$	$\hbar/2(-i|0\rangle\langle 1| + i|1\rangle\langle 0|)$	$\hbar/2(-i|z_+\rangle\langle z_-| + i|z_-\rangle\langle z_+|)$
S_z	$\mathbf{S_z}$	$\hbar/2\ \mathbb{Z}$	$\hbar/2(|z_+\rangle\langle z_+| - |z_-\rangle\langle z_-|)$	$\hbar/2(|0\rangle\langle 0| - |1\rangle\langle 1|)$	$\hbar/2(|z_+\rangle\langle z_+| - |z_-\rangle\langle z_-|)$

Appendix 5: Probability as a function of measurement angle

Consider a particle with a "definite" spin axis in the xz-plane. By "definite" spin axis we mean that a measurement produces $+\hbar/2$ with probability 1 in one direction and produces $-\hbar/2$ with probability 1 in the direction 180° from the first.
Now assume that a measurement is made at an angle α from the spin-up, i.e., z-axis. as shown in Fig. 4 below.

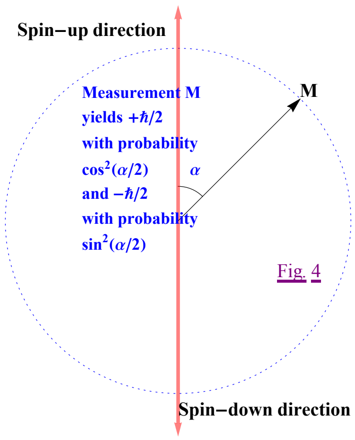


Fig. 4

Then the probability of $+\hbar/2$ is $\mathrm{Cos}^2[\alpha/2]$ and of $-\hbar/2$ is $\mathrm{Sin}^2[\alpha/2]$.
Proof:
The operator associated with this measurement (or observable) is $\boldsymbol{S}_\alpha = \cos(\alpha)\,\boldsymbol{S}_z + \sin(\alpha)\,\boldsymbol{S}_x$. Since $\boldsymbol{S}_x = \frac{\hbar}{2}\boldsymbol{X}$ and $\boldsymbol{S}_z = \frac{\hbar}{2}\boldsymbol{Z}$, we have $\boldsymbol{S}_\alpha = \frac{\hbar}{2}(\sin(\alpha)\,\boldsymbol{X} + \cos(\alpha)\,\boldsymbol{Z})$. Recalling that the matrix representations X and Z of the Pauli operators $\boldsymbol{X}$ and $\boldsymbol{Z}$ are given by $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, respectively, we have that the matrix representation for $\boldsymbol{S}_\alpha$ is

$S_\alpha = \hbar/2\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$. Thus define:

```
Sα = ħ / 2 {{Cos[α], Sin[α]}, {Sin[α], -Cos[α]}};
```

Then one set of eigenvectors of S_α is:

```
e1 = {Cos[α / 2], Sin[α / 2]};
e2 = {-Sin[α / 2], Cos[α / 2]};
Map[MatrixForm, {e1, e2}]
```

$$\left\{\begin{pmatrix} \mathrm{Cos}\left[\frac{\alpha}{2}\right] \\ \mathrm{Sin}\left[\frac{\alpha}{2}\right] \end{pmatrix}, \begin{pmatrix} -\mathrm{Sin}\left[\frac{\alpha}{2}\right] \\ \mathrm{Cos}\left[\frac{\alpha}{2}\right] \end{pmatrix}\right\}$$

Note that they are orthonormal:

```
{e1.e2, e1.e1, e2.e2} // Simplify
{0, 1, 1}
```

Check that these are indeed eigenvectors associated with $+\hbar/2$ and $-\hbar/2$.

```
Sα.e1 // Simplify
Sα.e2 // Simplify
```

$\left\{\frac{1}{2}\hbar\,\mathrm{Cos}\left[\frac{\alpha}{2}\right], \frac{1}{2}\hbar\,\mathrm{Sin}\left[\frac{\alpha}{2}\right]\right\}$

$\left\{\frac{1}{2}\hbar\,\mathrm{Sin}\left[\frac{\alpha}{2}\right], -\frac{1}{2}\hbar\,\mathrm{Cos}\left[\frac{\alpha}{2}\right]\right\}$

Define the projection matrices (operators) onto the eigenspaces (rays) defined by the eigenvectors. This employs the outer product of each of the two eigenvectors with themselves.

```
project1 = Outer[Times, e1, e1];
project2 = Outer[Times, e2, e2];
Map[MatrixForm, {project1, project2}]
```

$$\left\{\begin{pmatrix} \mathrm{Cos}\left[\frac{\alpha}{2}\right]^2 & \mathrm{Cos}\left[\frac{\alpha}{2}\right]\mathrm{Sin}\left[\frac{\alpha}{2}\right] \\ \mathrm{Cos}\left[\frac{\alpha}{2}\right]\mathrm{Sin}\left[\frac{\alpha}{2}\right] & \mathrm{Sin}\left[\frac{\alpha}{2}\right]^2 \end{pmatrix}, \begin{pmatrix} \mathrm{Sin}\left[\frac{\alpha}{2}\right]^2 & -\mathrm{Cos}\left[\frac{\alpha}{2}\right]\mathrm{Sin}\left[\frac{\alpha}{2}\right] \\ -\mathrm{Cos}\left[\frac{\alpha}{2}\right]\mathrm{Sin}\left[\frac{\alpha}{2}\right] & \mathrm{Cos}\left[\frac{\alpha}{2}\right]^2 \end{pmatrix}\right\}$$

Check that they act like projections on the eigenvector basis:

```
project1.e1 // Simplify
project2.e2 // Simplify
project1.e2 // Simplify
project2.e1 // Simplify
```

$\left\{\mathrm{Cos}\left[\frac{\alpha}{2}\right], \mathrm{Sin}\left[\frac{\alpha}{2}\right]\right\}$

$\left\{-\mathrm{Sin}\left[\frac{\alpha}{2}\right], \mathrm{Cos}\left[\frac{\alpha}{2}\right]\right\}$

```
{0, 0}
{0, 0}
```

Now check that $S_\alpha = \hbar/2\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ is a linear combination of the projections, with the eigenvalues as coefficients:

```
ℏ / 2 project1 - ℏ / 2 project2 // FullSimplify // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2}\hbar\,\mathrm{Cos}[\alpha] & \frac{1}{2}\hbar\,\mathrm{Sin}[\alpha] \\ \frac{1}{2}\hbar\,\mathrm{Sin}[\alpha] & -\frac{1}{2}\hbar\,\mathrm{Cos}[\alpha] \end{pmatrix}$$

Finally, the Hilbert space story of QM tells us that the observable S_α has the eigenvalues $+\hbar/2$ and $-\hbar/2$ as possible numerical outcomes, and that the associated probabilities are given by $|\langle e_1|\psi\rangle|^2$ and $|\langle e_2|\psi\rangle|^2$, respectively, where e_1 and e_2 are from the (orthonormal!) eigenvector basis given above and $|\psi\rangle$ is the state of the system.

Referring way back to Figure 4 at the beginning of this section, we see that the state $|\psi\rangle$ is $|z_+\rangle$ or $|\mathrm{up}\rangle$, or however you want to name the spin-up state along the z-axis. In the standard z-basis, the corresponding vector is $z_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore, we have the probabilities of getting $+\hbar/2$ and $-\hbar/2$ are, respectively (drum roll, please!):

```
z₊ = {1, 0};
(e1.z₊)^2
(e2.z₊)^2
```

$\text{Cos}\left[\frac{\alpha}{2}\right]^2$

$\text{Sin}\left[\frac{\alpha}{2}\right]^2$

Note, as expected, that the probabilities are reversed if the system is in the spin-down state (with associated vector $z_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$):

```
z₋ = {0, 1};
(e1.z₋)^2
(e2.z₋)^2
```

$\text{Sin}\left[\frac{\alpha}{2}\right]^2$

$\text{Cos}\left[\frac{\alpha}{2}\right]^2$

Appendix 6: Entangled qubit systems (part 1)

- **Assume we have a source that produces two qubit systems A and B in an entangled singlet state $\left|\Psi_-^{(AB)}\right\rangle = (1/\sqrt{2})\,(\,|0, 1\rangle - |1, 0\rangle\,)$. (AB) denotes the composite system.**
  - ***Note tensor products are involved here (see Appendix 1)! For example $|a, b\rangle \doteq |a\rangle \otimes |b\rangle$*** **whenever $|a\rangle$ is a state in system A and $|b\rangle$ is a state in system B.**
  - **This means $\left|\Psi_-^{(AB)}\right\rangle$ *cannot* be expressed as a simple tensor product of states (vectors), one from A and one from B. *It is not possible to assign individual states to the individual subsystems.***
- **The A qubit is delivered to Alice and the B qubit is delivered to Bob.**
- **Denote $\left|\Phi_+^{(AB)}\right\rangle = (1/\sqrt{2})\,(\,|0, 0\rangle + |1, 1\rangle\,)$**
- **Recall the Pauli operators $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $X = |0\rangle\langle 1| + |1\rangle\langle 0|$.**
- **For operators $X^{(A)}$ and $Z^{(B)}$ on a composite system AB, the tensor product can be displayed in either of the two following notations: $X^{(A)} Z^{(B)} \doteq X^{(A)} \otimes Z^{(B)}$.**
- **Then if X_A, X_B, Z_A, Z_B denote the Pauli operators associated with observables along the X and Z axes for the A and B qubit systems, we have:**
  - $Z^{(A)} Z^{(B)} \left|\Psi_-^{(AB)}\right\rangle = -\left|\Psi_-^{(AB)}\right\rangle$
    $X^{(A)} X^{(B)} \left|\Psi_-^{(AB)}\right\rangle = -\left|\Psi_-^{(AB)}\right\rangle$
    $X^{(A)} Z^{(B)} \left|\Psi_-^{(AB)}\right\rangle = -\left|\Phi_+^{(AB)}\right\rangle$
    $Z^{(A)} X^{(B)} \left|\Psi_-^{(AB)}\right\rangle = +\left|\Phi_+^{(AB)}\right\rangle$

Proof: We do the first one as an example. The others are similar. First we note:
$Z\,|0\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)\,|0\rangle = |0\rangle\langle 0|\,|0\rangle - |1\rangle\langle 1|\,|0\rangle = \langle 0\,|0\rangle\,|0\rangle - \langle 1\,|0\rangle\,|1\rangle = |0\rangle$, and similarly,
$Z\,|1\rangle = -|1\rangle$.
$X\,|0\rangle = |1\rangle$ and
$X\,|1\rangle = |0\rangle$.

Thus

$$Z^{(A)} Z^{(B)} \left|\Psi_-^{(AB)}\right\rangle = \frac{1}{\sqrt{2}} Z^A \otimes Z^{(B)} \left(\left|0, 1\right\rangle - \left|1, 0\right\rangle\right) = \frac{1}{\sqrt{2}}\left(Z^A \otimes Z^{(B)}\left|0, 1\right\rangle - Z^A \otimes Z^{(B)}\left|1, 0\right\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\left(Z^A\left|0\right\rangle \otimes Z^{(B)}\left|1\right\rangle - Z^A\left|1\right\rangle \otimes Z^{(B)}\left|0\right\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\left(-\left|0\right\rangle \otimes \left|1\right\rangle + \left|1\right\rangle \otimes \left|0\right\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\left(-\left|0, 1\right\rangle + \left|1, 0\right\rangle\right) = -\left|\Psi_-^{(AB)}\right\rangle$$

Appendix 7: Entangled qubit systems (part 2)

The expectation $\langle \boldsymbol{X} \rangle$ of an operator $\boldsymbol{X}$ is defined by $\langle \boldsymbol{X} \rangle = \langle \psi | \, \boldsymbol{X} \, | \psi \rangle$ if the system is in state ψ). See Appendix 1.
From equations (1) above, we can then derive expectations for the (tensor) products of operators on a composite system (i.e., tensor product of Hilbert spaces). We assume the composite system is a (tensor) product of two qubit systems, and in the (entangled) state $\left|\Psi_-^{(\mathrm{AB})}\right\rangle = \frac{1}{\sqrt{2}} \left(\left| 0, 1 \right\rangle - \left| 1, 0 \right\rangle \right)$.

Then we have:
$\left\langle \boldsymbol{Z}^{(A)} \boldsymbol{Z}^{(B)} \right\rangle = -1$
$\left\langle \boldsymbol{X}^{(A)} \boldsymbol{X}^{(B)} \right\rangle = -1$
$\left\langle \boldsymbol{X}^{(A)} \boldsymbol{Z}^{(B)} \right\rangle = 0$
$\left\langle \boldsymbol{Z}^{(A)} \boldsymbol{X}^{(B)} \right\rangle = 0$

<u>Proof</u>: We do two examples. The others are similar.
$\left\langle \boldsymbol{Z}^{(A)} \boldsymbol{Z}^{(B)} \right\rangle = \left\langle \Psi_-^{(\mathrm{AB})} \left| \boldsymbol{Z}^{(A)} \boldsymbol{Z}^{(B)} \right| \Psi_-^{(\mathrm{AB})} \right\rangle = -\left\langle \Psi_-^{(\mathrm{AB})} \right| \left(\boldsymbol{Z}^{(A)} \boldsymbol{Z}^{(B)} \left| \Psi_-^{(\mathrm{AB})} \right\rangle \right) = -\left\langle \Psi_-^{(\mathrm{AB})} \middle| \Psi_-^{(\mathrm{AB})} \right\rangle = -1$
$\langle \boldsymbol{X}^{(A)} \boldsymbol{Z}^{(B)} \rangle = \left\langle \Psi_-^{(\mathrm{AB})} \right| \boldsymbol{X}^{(A)} \boldsymbol{Z}^{(B)} - \left| \Psi_-^{(\mathrm{AB})} \right\rangle = \left\langle \Psi_-^{(\mathrm{AB})} \right| \left(\boldsymbol{X}^{(A)} \boldsymbol{Z}^{(B)} \left(- \left| \Psi_-^{(\mathrm{AB})} \right\rangle \right) \right) =$
$-\left\langle \Psi_-^{(\mathrm{AB})} \middle| \Phi_+^{(\mathrm{AB})} \right\rangle = 0$

$\left\langle \Psi_-^{(\mathrm{AB})} \middle| \Phi_+^{(\mathrm{AB})} \right\rangle = 0$ (for example) because -- (remember the basis $\{|0\rangle_A, |1\rangle_A\}$ is orthonormal for A and $\{|0\rangle_B, |1\rangle_B\}$ is orthonormal for B) -- we suppress the subscripts in the following):
$\left\langle \Psi_-^{(\mathrm{AB})} \middle| \Phi_+^{(\mathrm{AB})} \right\rangle = \left\langle \frac{1}{\sqrt{2}} \left(\left| 0, 1 \right\rangle - \left| 1, 0 \right\rangle \right) \middle| \frac{1}{\sqrt{2}} \left(\left| 0, 0 \right\rangle + \left| 1, 1 \right\rangle \right) \right\rangle = \frac{1}{2} \left\langle \left| 0, 1 \right\rangle \middle\| 0, 0 \right\rangle \rangle + \frac{1}{2} \left\langle \left| 0, 1 \right\rangle \| 1, 1 \right\rangle \rangle -$
$\frac{1}{2} \left\langle \left| 1, 0 \right\rangle \middle\| 0, 0 \right\rangle \rangle - \frac{1}{2} \left\langle \left| 1, 0 \right\rangle \| 1, 1 \right\rangle \rangle =$
$\frac{1}{2} \left(\langle 0 | 0 \rangle \langle 1 | 0 \rangle + \langle 0 | 1 \rangle \langle 1 | 1 \rangle - \langle 1 | 0 \rangle \langle 0 | 0 \rangle - \langle 1 | 1 \rangle \langle 0 | 1 \rangle \right) = 0$

Initialization

```
circle = Graphics[{Blue, Dotted, Circle[{0, 0}, 1]}];
arc1 = Graphics[Circle[{0, 0}, .2, {-45 Degree, 45 Degree}]];
arc2 = Graphics[Circle[{0, 0}, .2, {45 Degree, 90 Degree}]];
arc3 = Graphics[Circle[{0, 0}, .2, {-45 Degree, -90 Degree}]];
arc4 = Graphics[Circle[{0, 0}, .2, {-45 Degree, -135 Degree}]];
arc5 = Graphics[Circle[{0, 0}, .2, {-90 Degree, -135 Degree}]];
spinup = Graphics[{Red, Opacity[0.5], Thickness[0.01], Arrow[{{0, 0}, {0, 1.25}}]}];
spindown = Graphics[{Red, Opacity[0.5], Thickness[0.01], Arrow[{{0, 0}, {0, -1.25}}]}];
aMeasure = Graphics[{Arrow[{{0, 0}, {Cos[45 Degree], Sin[45 Degree]}}]}];
bMeasure = Graphics[{Arrow[{{0, 0}, {Cos[90 Degree], Sin[90 Degree]}}]}];
cMeasure = Graphics[{Arrow[{{0, 0}, {Cos[-90 Degree], Sin[-90 Degree]}}]}];
dMeasure = Graphics[{Arrow[{{0, 0}, {Cos[-45 Degree], Sin[-45 Degree]}}]}];

spin45 = Graphics[{Red, Opacity[0.5], Thickness[0.01],
    Arrow[{{0, 0}, 1.25 {Cos[45 Degree], Sin[45 Degree]}}]}];
spinm45 = Graphics[{Red, Opacity[0.5], Thickness[0.01],
    Arrow[{{0, 0}, 1.25 {Cos[-135 Degree], Sin[-135 Degree]}}]}];
measure10 = Graphics[{Arrow[{{0, 0}, {Cos[80 Degree], Sin[80 Degree]}}]}];
measure25 = Graphics[{Arrow[{{0, 0}, {Cos[65 Degree], Sin[65 Degree]}}]}];
measure145 = Graphics[{Arrow[{{0, 0}, {Cos[-55 Degree], Sin[-55 Degree]}}]}];
measure170 = Graphics[{Arrow[{{0, 0}, {Cos[-80 Degree], Sin[-80 Degree]}}]}];
measurem135 = Graphics[{Arrow[{{0, 0}, {Cos[-135 Degree], Sin[-135 Degree]}}]}];
arc10 = Graphics[Circle[{0, 0}, .2, {90 Degree, 80 Degree}]];
arc25 = Graphics[Circle[{0, 0}, .2, {80 Degree, 65 Degree}]];
arc145 = Graphics[Circle[{0, 0}, .2, {65 Degree, -55 Degree}]];
arc170 = Graphics[Circle[{0, 0}, .2, {-55 Degree, -80 Degree}]];

rightPointer = Graphics[{Blue, Arrowheads[0.1], Thick, Arrow[{{-.5, 0}, {.5, 0}}]}];
leftPointer = Graphics[{Blue, Arrowheads[0.1], Thick, Arrow[{{.8, 0}, {-.8, .5}}]}];
SetOptions[EvaluationNotebook[], ShowGroupOpener → True]
```