Hilbert Space for Quantum Mechanics
The Remaining Big Theorems

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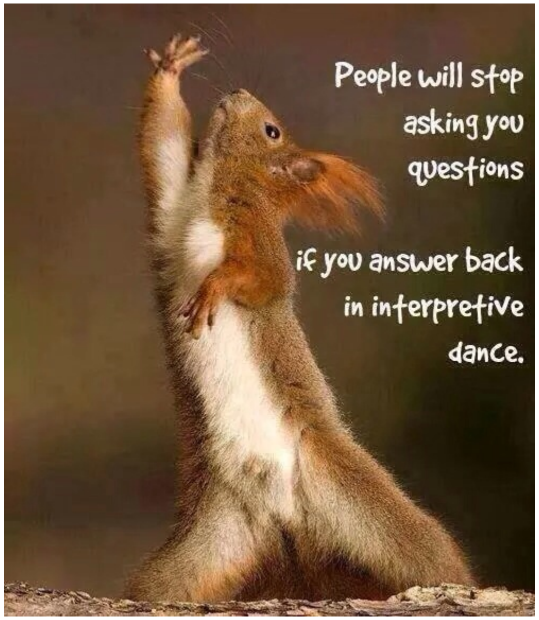

Multiplication Operator I

If μ is a finite measure on measure space M and $f: M \to \mathbb{R}$ is measurable define

$$\mathcal{D}_T = \{\, g \in \mathcal{L}^2(M,\mu) \mid fg \in \mathcal{L}^2(M,\mu) \,\}$$

and $T\colon \mathcal{D}_T \to \mathcal{L}^2(M,\mu)$ by $T(g) = fg$.

Then T is self-adjoint and $\sigma(T)$ is the essential range of f,

$$\sigma(T) = \{\, \lambda \in \mathbb{R} \mid \mu\,(\,\{\, m \in M \mid \lambda - \varepsilon < f(m) < \lambda + \varepsilon \,\}) > 0 \,\}.$$

If there are various operators like this in play we will call this one T_f.

Multiplication Operator II

Suppose S is any self-adjoint operator on Hilbert space $\mathcal{H}$. Then there is a set M with finite measure μ and an isometry $U\colon \mathcal{H} \to \mathcal{L}^2(M,\mu)$ onto its range and a measurable function $f\colon M \to \mathbb{R}$ so that

(i) $\psi \in \mathcal{D}_S$ if and only if $f\,U(\psi) \in \mathcal{L}^2(M,\mu)$.

(ii) If $g \in U(\mathcal{D}_S)$ then $(USU^{-1}g)(m) = f(m)g(m)$ for all $m \in M$. (a.e. wrt μ)

Functional Calculus

- If S is any self-adjoint operator on Hilbert space $\mathcal{H}$ there is a unique map ϕ from B, the bounded Borel functions on $\mathbb{R}$ to $\mathfrak{CL}(\mathcal{H})$ with the following properties.
- (i) ϕ is a $*$-algebra homomorphism.
- (ii) ϕ is norm-continuous. In fact, $\|\phi(h)\| \leq \|h\|_\infty \; \forall h \in B$.
- (iii) If $h_n \to x \in B$ then $\phi(h_n)(\psi) \to S\psi$ on $\mathcal{D}_S$.
- (iv) If $h_n(x) \to h(x)$ for each x and $\|h_n\|_\infty$ is bounded then $\phi(h_n) \to \phi(h)$ strongly.

Functional Calculus II

Given all these properties it makes sense to denote $\phi(h)$ by $h(S)$.

(i)-(iv) imply, additionally, that:

- (v) If $S\psi = \lambda\psi$ then $\phi(h)\psi = h(\lambda)\psi$.
- (vi) If $h \geq 0$ then $\phi(h) \geq 0$.

Projection-valued Measure I

If S is any self-adjoint operator on Hilbert space $\mathcal{H}$ and χ_A is the characteristic function of Borel $A \subset \mathbb{R}$ define $P_A = \phi(\chi_A) = \chi_A(S)$. And define $P_t = P_{(-\infty,t]}$.

- Since $\chi_A^2 = \chi_A$ and χ_A is real valued each P_A is an orthogonal projection, $P_\varnothing = 0$ and $P_\mathbb{R} = I$, the identity on $\mathcal{H}$.
- $P_A P_B = P_{A\cap B}$.
- (i) If $A = \bigcup_{i=1}^\infty$ and $A_i \cap A_j = \varnothing$ whenever $i \neq j$ then P_A is the strong limit of $\sum_{i=1}^n P_{A_i}$.

Projection-valued Measure II

- For each fixed ψ the inner product $\langle \psi, P_A(\psi) \rangle$ is an ordinary measure on $\mathbb{R}$ and we use $d\langle \psi, P_A(\psi) \rangle$ to denote integration with respect to that measure.
- For each ψ the map

$$f \to \int f(t) d\langle \psi, P_t(\psi) \rangle$$

is bounded and linear, and so corresponds to inner product against some $\eta = B\psi$, as in $\langle \psi, B\psi \rangle = \int f(t) d\langle \psi, P_t(\psi) \rangle$.
- We denote this B by $B = \int f(t) dP_t$ and show that $S = \int t\, dP_t$.

Projection-valued Measure III

Theorem

There is a one-to-one correspondence between self-adjoint operators T and projection-valued measures P given by

$$S \longrightarrow P$$

$$P \longrightarrow \int t\, dP_t.$$

Stone's Theorem I

- A strongly continuous one-parameter group of unitary operators on Hilbert $\mathcal{H}$ is a map $U\colon \mathbb{R} \to \mathcal{CL}(\mathcal{H})$ which satisfies
  (i) $U(t)$ is unitary and $U(s+t) = U(s)U(t)$ for $s, t \in \mathbb{R}$.
  (ii) If $t_i \to a$ then $U(t_i) \to U(a)$ strongly.
- If T is self-adjoint define $U(t) = e^{itT}$. Then U is a strongly continuous one-parameter group of unitary operators and, in addition,
  (iii) For each $\psi \in \mathcal{D}_T$ we have $\frac{U(t)\psi - \psi}{t} \to iT\psi$ as $t \to 0$ and
  (iv) If $\lim_{t\to 0} \frac{U(t)\psi - \psi}{t}$ exists then $\psi \in \mathcal{D}_T$.

Stone's Theorem II

Theorem

If U is any strongly continuous one-parameter group of unitary operators then there is a self-adjoint operator T with $U(t) = e^{itT}$.

T is called the infinitesimal generator of U.

The phrasing of these results, and their order, was adapted from the superb standard reference by Reed and Simon, *Functional Analysis*, Academic Press, 1980. I ***highly*** recommend it as a very clean mathematical presentation with pedagogical flair, filled with insight into the sticky issues involved in applications to quantum physics.