# Hilbert Space for Quantum Mechanics Spectral Theorems for Normal (Bounded) Operators

Larry Susanka

Mathematics Department
Bellevue College

June 5, 2014



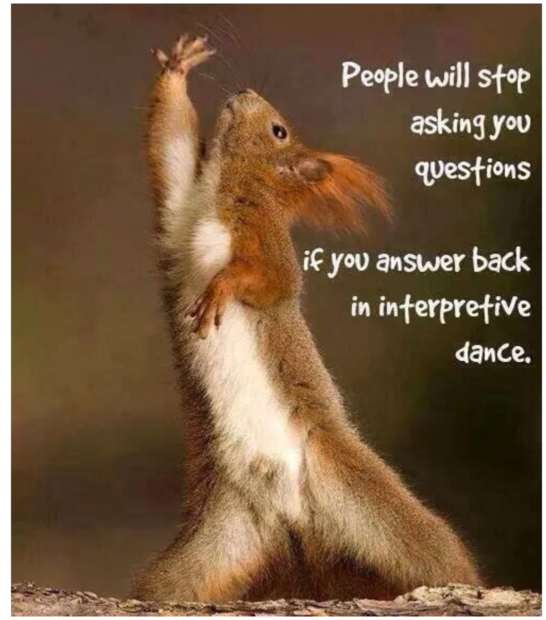




Table of Contents





## The Resolvent

- We suppose $T$ to be a bounded operator on $\mathcal{H}$

  The resolvent set for $T$ is

  $$\rho(T) = \{\, \lambda \mid T - \lambda I \text{ is one-to-one and onto} \,\}.$$

- By the Closed Graph Theorem the resolvent function

  $$R_\lambda = (T - \lambda I)^{-1}$$

  is continuous whenever $\lambda \in \rho(T)$: $T - \lambda I$ is bounded below and, specifically, if $k = \|R_\lambda\|$ then

  $$k\,\|x\| \leq \|(T - \lambda I)(x)\| \leq \frac{\|x\|}{k} \quad \forall x \in \mathcal{H}.$$

# The Spectrum

- The spectrum for $T$ is

  $\sigma(T) = \mathbb{C} - \rho(T) = \{\, \lambda \mid T - \lambda I \text{ is not one-to-one or not onto} \,\}.$

- If $T - \lambda I$ is not one-to-one $\lambda$ is called an eigenvalue and $Ker(T - \lambda I)$ is called the eigenspace for $\lambda$.
- If $T$ is normal and $T - \lambda I$ is one-to-one then $T - \lambda I$ must be onto. So $\mathcal{R}_{T-\lambda I} = \mathcal{H}$ unless $\lambda$ is an eigenvalue.

  (For a normal operator $\mathcal{R}_{T-\lambda I} = Ker(T - \lambda I)^{\perp}$.)



# The Spectrum II

- Theorem: The spectrum for $T$ is closed, bounded and nonempty. The proof relies on the following....
- bounded: If $|\lambda| > \|T\|$ then $\lambda \in \rho(T)$.
- If $A$ is a bounded operator and $\|A\| < 1$ then

  $$(I - A)^{-1} = I + A + A^2 + \cdots$$

  converges in operator norm. Such limits are continuous since the space of bounded operators is Banach. So

  $$R_\lambda = (T - \lambda I)^{-1} = -\frac{1}{\lambda}\left(I + \frac{T}{\lambda} + \frac{T^2}{\lambda^2} + \cdots\right)$$

  also converges in operator norm.



# The Spectrum III

- So if $\mu$ is very near $\lambda$ then $\frac{1}{\mu-\lambda}$ is huge so

  $$\left(\sum_{m=0}^{\infty} (\mu - \lambda)^m R_\lambda^m\right) R_\lambda$$

  converges in operator norm. Multiplying the sum by $T - \mu I$ yields (eventually) the conclusion that this product is $I$. So the large sum is $R_\mu$, which is therefore continuous. So $\rho(T)$ is open.
- If $x, y \in \mathcal{H}$ the function $M(\mu) = \langle R_\mu x \,,\, y \rangle$ is holomorphic: given by a complex power series.
- Also $R_\mu = (T - \mu I)^{-1} \to 0$ as $\mu \to \infty$. So if $\sigma(T) = \varnothing$ we have (By Liouvilles's Theorem) $M(\mu)$ constant, and that constant must be 0, and this is true for every $x$ and $y$. So $R_\mu = 0$.
- Since $R_\mu(T - \mu I) = I$ this is impossible.



# The Spectrum IV

- The spectrum of a normal operator has two parts.

  $\sigma_p(T)$ consisting of all the eigenvalues and the continuous spectrum $\sigma_c$: the rest.

- If $\lambda \in \sigma(T)$ then there is a sequence $x_n$ of unit vectors for which $(T - \lambda I)x_n \to 0$.
- If $T$ is normal, $\sigma(T^\dagger) = \overline{\sigma(T)}$.
- If $T$ is self-adjoint, $\sigma(T) \subset \mathbb{R}$.
- If $T$ is unitary, $\sigma(T) \subset \mathbb{S}^1$, the unit circle in $\mathbb{C}$.