Hilbert Space Methods Used in a First Course in Quantum Mechanics

Topics Concerning Metric Spaces

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Definition

A **pseudometric space** is a pair (V, d) where V is a nonempty set and

$$d\colon V \times V \to [0, \infty)$$

is a function called a **pseudometric** or **distance function** which has (by definition) the following properties:

- $d(x,x) = 0 \;\forall x \in V$.
- $d(x,y) = d(y,x) \;\forall x, y \in V$ (symmetry).
- $d(x,z) \le d(x,y) + d(y,z) \;\forall x, y, z \in V$ (triangle inequality).

If, in addition, $d(x,y) = 0$ implies $x = y$ the pseudometric is called a **metric.**

A nonempty subset S of pseudometric space V is called **bounded** if there is a real number c for which $d(x,y) \le c$ for each $x, y \in S$.

The First Few

Consider $V = \mathbb{R}$ with $d(x,y) = |\,x - y\,|$.

Now let V be $\mathbb{R}^2$ and define d by

a. $d(\,(x,y),\,(v,w)\,) = \sqrt{(x-v)^2 + (y-w)^2}$.

b. or $d(\,(x,y),\,(v,w)\,) = |x - v| + |y - w|$.

c. or $d(\,(x,y),\,(v,w)\,) = \sup\{\,|x-v|,\,|y-w|\,\}$.

Now try the same for $\mathbb{R}^n$.

Sequence Examples

Let $V = \mathbb{R}^{\mathbb{N}}$, the set of sequences of real numbers with typical element $\mathbf{x} = (x_n)$.

- Confine attention to those $\mathbf{x}$ for which $\sum_{n=0}^{\infty} x_n^2 < \infty$. Define $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{n=0}^{\infty} (x_n - y_n)^2}$.
- Now consider only those $\mathbf{x}$ for which $\sum_{n=0}^{\infty} |x_n| < \infty$. Define $d(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^{\infty} |x_n - y_n|$.
- Finally, for bounded sequences define

$$d(\mathbf{x}, \mathbf{y}) = \sup \{ |x_n - y_n| \mid n \in \mathbb{N} \} .$$

Function Spaces

Let $V = \mathbb{R}^{[0,1]}$, the set of real functions on the unit interval.

- Confine attention to those members $\mathbf{f}$ of V for which $\int_0^1 \mathbf{f}(x)^2 \, dx < \infty$. Define

$$d(\mathbf{f}, \mathbf{g}) = \sqrt{\int_0^1 (\mathbf{f}(x) - \mathbf{g}(x))^2 \, dx}.$$

- Now consider only those $\mathbf{f}$ for which $\int_0^1 |\mathbf{f}(x)| < \infty$. Define

$$d(\mathbf{f}, \mathbf{g}) = \int_0^1 |\mathbf{f}(x) - \mathbf{g}(x)| \, dx.$$

- Finally, for bounded functions define

$$d(\mathbf{f}, \mathbf{g}) = \sup \{ |\mathbf{f}(x) - \mathbf{g}(x)| \mid x \in [0,1] \} .$$

- Take the same pseudometric as the last but restrict attention to continuous functions only. This space is denoted $C[0,1]$.

Stranger Examples (1)

If V is any set we can let $d(x,y) = 0$ when $x = y$ and $d(x,y) = 1$ otherwise. This is called the **discrete metric**.

For a more interesting example we can let

$$V = ([0,1) \times \{ 0 \}) \bigcup \{ (1,1), \ (1,-1) \}.$$

Now define $d((x,0), \ (v,0)) = |x - v|$ and $d((x,0), \ (1,w)) = 1 - x$ and $d((1,y), \ (1,w)) = 0$.

This makes V into a pseudometric space.

We can make other pseudometrics with interesting features on this space too.

Stranger Examples (2)

When $V = \mathbb{R}^2$ we can define

$$d((x,y), \ (v,w)) = \begin{cases} \sqrt{(x-v)^2 + (y-w)^2}, & \text{if } xw = yv; \\ \sqrt{x^2 + y^2} + \sqrt{v^2 + w^2}, & \text{otherwise.} \end{cases}$$

$\mathbb{R}^2$ is a metric space with this distance function.

Definitions

We presume as given some pseudometric space (V, d).

An **open ball of radius r centered at v** is the collection of all members of V whose distance from v is less than r.

This open ball is denoted $\mathbf{B}(\mathbf{v}, \mathbf{r})$.

A **closed ball of radius r centered at v** is the collection of all members of V whose distance from v does not exceed r.

This closed ball is denoted $\mathbf{C}(\mathbf{v}, \mathbf{r})$.

A **sphere of radius r centered at v** is the collection of all members of V whose distance from v is r.

This sphere is denoted $\mathbf{S}(\mathbf{v}, \mathbf{r})$.

Note $\mathbf{C}(\mathbf{v}, \mathbf{r}) = \mathbf{B}(\mathbf{v}, \mathbf{r}) \cup \mathbf{S}(\mathbf{v}, \mathbf{r})$.

Open and Closed

We presume as given some pseudometric space (V, d).

A subset S of V is called **open in V** if it is the union of open balls. *topology*

A subset S of V is called **closed in V** if $V - S$ is open. *topology*

A point x in V is said to be a **limit point of a subset S of V** if every open set containing x contains at least one member of S other than x itself. (Note: We do not require x to be in S, but it might be.) *topology*

An alternative definition of closed set: S is closed exactly when it contains all of its limit points. *analysis*

The **closure of a set S** consists of S together with all its limit points. This set is denoted $\overline{S}$.

Is it true that $C(v, r) = \overline{B(v, r)}$?

Continuity

We presume as given pseudometric spaces (V, d) and (W, d).

Note the pseudometric is denoted by the same symbol, d for both domain and range. The one intended is derived from context. It is rarely an issue.

A sequence (x_n) in V or W is said to **converge to L**, written $x_n \to L$, when the numerical sequence $d(x_n, L)$ converges to 0.

A function $F\colon V \to W$ is called **continuous at the point $\mathbf{v} \in \mathbf{V}$** provided

$$F(v_n) \to F(L) \quad \text{whenever} \quad v_n \to L.$$

The function is called **continuous** if it is continuous at every point in its domain.

Uniform Continuity

The definition of continuity given above is equivalent to the more familiar form, usually given as

$$\forall r > 0\ \exists t > 0 \text{ so that } d(F(w), F(v)) < r \text{ whenever } d(w, v) < t.$$

Continuity means that points near each other map to points near each other. Small balls map to small balls. However *near* and *small* can mean different things at different places in V.

The function is called **uniformly continuous** if t can be chosen independently of the point v: that is, the same t *works* for a given r for every v.

Examples of *uniform continuity* versus *not*...

The Following Are Equivalent

TFAE:

F is continuous.

$F^{-1}(\mathcal{O})$ is open in V whenever $\mathcal{O}$ is open in W.

$F^{-1}(\mathcal{C})$ is closed in V whenever $\mathcal{C}$ is closed in W.

Cauchy Sequences and Completeness

A sequence (x_n) in pseudometric V is called **Cauchy** if for every $r > 0$ there is an integer k so that $d(x_n, x_m) < r$ whenever both m and n are beyond k.

A Cauchy sequence must be bounded: the tail of the sequence fits into a ball, and the beginning is finite, so it all fits in a ball.

If $x_n \to L$ then if m and n are both large both x_n and x_m are close to L, so they must be close to each other. So ...

Every convergent sequence is Cauchy.

A metric space V is called **complete** if every Cauchy sequence in V converges to a member of V.

Compactness

A subset C of pseudometric V is called **compact** if every sequence in C has a subsequence which converges to a member of C. *analysis*

A topologist, however, says that a set is compact if every open cover has a finite subcover. The two definitions are equivalent in a pseudometric space.

Every limit point has a sequence that converges to it, and compactness requires that point to be in C, so...

In a metric space **compact sets are closed.**

Are closed sets compact? In what context?

Bolzano-Weierstass

Bolzano-Weierstrass Theorem

In a finite dimensional Euclidean space, every closed and bounded set is compact.

What about the infinite dimensional case that concerns us here?

Recall the sequence space $\mathbb{R}^{\mathbb{N}}$ with metric given by the sum of the squared differences of the sequence elements.

Consider the sequence given by

$$e_1 = (1, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0, \ldots), \quad e_3 = (0, 0, 1, 0, \ldots), \quad \ldots$$

The closed ball centered at the zero sequence and with radius 1, which we will call C here, contains all the e_n.

Is C closed and bounded?

Does (e_n) have any convergent subsequence?

Baire Category

A subset S of V is called **dense** if every member of V is a limit point of S.

V is called **separable** if it contains a countable dense subset.

Every compact metric space is separable, and more! Think about it ...

The Baire Category Theorem

If V is a complete metric space every countable collection of open dense subsets has dense intersection.