A Few Details of the History of Functional Analysis

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Overview

Modern Functional Analysis, the mathematical underpinnings of what we are trying to understand here, can be thought of as a blending of Topology and Algebra, and sometimes Geometry too.

The topological part is concerned with *nearness* and convergence, and the algebraic part refers to Linear Algebra adapted to vector spaces without finite basis. Geometry refers to understanding perpendicularity: sometimes we have that, sometimes we don't.

Overview

The combination of these in modern form took a long time to create, well over 50 years of continuous effort by many mathematicians. Virtually all of modern mathematics was involved—most of modern mathematics either had its roots there or has theorems directly pertinent to FA.

The effort can be described as the mature development of the concepts of duality (apparently an unknown idea before 1900) and spectral theory.

Here are a few of the major characters in this story.

JOSEPH FOURIER 1768-1830

Most pertinent publications 1807 and 1822,

concepts of convergence of functions,

the concept of *arbitrary function*,

growing awareness of different types of convergence,

birth of spectral theory

BERNHARD RIEMANN 1826-1866

In *On the Foundations which Underlie Geometry* (1854): he talks of *a finite dimensional multiplicity, where the position of a point is determined by a finite set of numbers* (finite dimensional manifold followed by curvature tensor)

He goes on to talk of *multiplicities* where a finite set of numbers will not suffice, giving as an example *all possible determinations of a function in a given domain.* Riemann is talking about infinite dimensional function spaces here.

Georg Cantor 1845-1914

Starts the debate, in its most direct form, about the meaning of infinite sets.

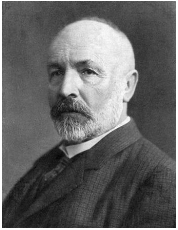
His immediate context was the exceptional sets of Fourier series (1869)

Treats infinite objects as first-class entities

Leads to an (at first) uneasy acceptance of an analogue to Euclidean space of infinite dimension

Poincaré referred to Cantor's ideas as a *grave disease infecting the discipline of mathematics*

Kronecker: he is *a scientific charlatan*, a *renegade* and a *corrupter of youth*



Hilbert: *No one shall expel us from the Paradise that Cantor has created.*

Hermann Schwarz 1842-1921, Henri Poincaré 1854-1912, Charles Picard 1856-1941

Hermann Schwarz' 1885 paper on *vibrating membranes*,

Henri Poincaré's work from 1890,

Fixed point iteration: Charles Picard 1890

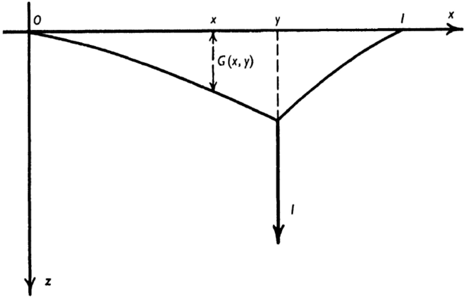
Vito Volterra 1860-1940, Erik Fredholm 1866 1927

Volterra used the term *functions of lines*, by which he meant functions *of functions* and worked on special cases of integral equations.

This work was shortly after greatly expanded upon by Erik Fredholm (1890) who gave a very general way of approaching integral equations.

The Spinning Shaft

From Tricomi's *Integral Equations* (1957): A spinning shaft rotating around the x axis and deformed under its own weight.



An elastic force acts to align the shaft; a centrifugal force acts to increase the deformation.

What rotation rate/deflection combinations can cause an equality between these two forces, both summed (integrated) over the whole length of the shaft?

The Spinning Shaft

Setting these two forces equal to each other, a simple (but clever) change of variable leads to the integral equation

$$\phi(x) = \lambda \int_0^1 K(x, y)\phi(y)\, dy$$

for constant λ, unknown *function* ϕ defined on $[0, 1]$ and *symmetric* kernel K defined on $[0, 1] \times [0, 1]$.

If we can find all such ϕ and λ combinations, we can invert the change of variable and produce the rotation rates and deflections we must avoid.

To our eyes, this *looks* like an eigenvalue equation.

Integral Equations

From this and many other such physical examples, Mathematicians (and Physicists and Engineers) were led to try to analyze, on a case-by-case basis, integral equations of the form

$$f(x) = \int_0^1 K(x,y)\phi(y)\,dy$$

where f and K are known functions and function ϕ is to be determined.

By analogy with Linear Algebra, if we could find a basis of eigenvectors for the linear function on the right we could diagonalize it, determine which functions f are in the range of the operator: the span of the eigenvectors for nonzero eigenvalues. There *is* a solution for these particular functions f.

Then we restrict to the orthogonal complement of the null space and invert the diagonal matrix on this subspace to locate the solution.

Integral Equations

But of course we can't use Linear Algebra for this. Linear Algebra is about (finite dimensional) vectors, and our linear operator is a function on functions or, as Volterra called it, a *function of lines*.

So Fredholm (and Volterra and many others before him) tried to convert this to a question involving Linear Algebra.

Their program, applied to the example above, was to use Riemann sum approximation with n terms, solve this finite dimensional problem, and then see what could be done about taking the limit as $n \to \infty$.

INTEGRAL EQUATIONS

Specifically, consider $f(x) = \int_0^1 K(x,y)\phi(y)\,dy$, called a Fredholm integral equation (of the second kind), and let

$$x_k = y_k = \frac{k}{n} \text{ for } k = 1, \ldots, n \text{ and } k_{j,k} = K(x_j, y_k) \text{ and } b_k = nf(x_k).$$

Letting z_k stand for the desired function value $\phi(y_k)$ we have

$$\begin{aligned}
b_1 &= k_{1,1}z_1 + k_{1,2}z_2 + \cdots + k_{1,n}z_n \\
b_2 &= k_{2,1}z_1 + k_{2,2}z_2 + \cdots + k_{2,n}z_n \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
b_n &= k_{n,1}z_1 + k_{n,2}z_2 + \cdots + k_{n,n}z_n
\end{aligned}$$

Integral Equations

Note that if K is symmetric, this system of equations has n eigenvectors for n real eigenvalues. If zero is not an eigenvalue there *will be* a solution.

Defining ϕ_n to be the function that is constant on each of the intervals of this Riemann sum,

$$\phi_n(x) = z_k \text{ for } x \in (x_{k-1}, x_k]$$

we have created a sequence of functions and a sequence of questions.

For instance, does this sequence converge? What about the eigenvalues? Can we do anything with the eigenvectors?

Integral Equations

Another type of integral equation, a Volterra equation (of the second kind):

$$f(x) = \int_0^x K(x,y)\phi(y)\,dy$$

corresponds to the above example where K is not symmetric but satisfies $K(x,y) = 0$ when $y > x$. Discretized, this can be solved by substitution:

$$\begin{aligned}
b_1 &= k_{1,1}z_1 + \quad 0 \quad + \quad 0 \quad + \cdots + \ 0 \\
b_2 &= k_{2,1}z_1 + k_{2,2}z_2 + \quad 0 \quad + \cdots + \ 0 \\
b_2 &= k_{3,1}z_1 + k_{3,2}z_2 + k_{3,3}z_2 + \cdots + \ 0 \\
&\ \ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
b_n &= k_{n,1}z_1 + k_{n,2}z_2 + k_{n,3}z_z \cdots + k_{n,n}z_n
\end{aligned}$$

DAVID HILBERT 1845-1914

David Hilbert, in communication with Fredholm and greatly interested in Mathematical Physics, immediately realized the significance of Fredholm's work.

In a few years he extracted the central concepts, axiomatized and greatly simplified them, and outlined a program for progress in the whole field.

Crystallization

According to Dieudonné in his *History of Functional Analysis* (1981) there was, between 1900 and 1910, a *a sudden crystallization of all the ideas and methods which had been slowly accumulating during the XIXth century …*

This was, essentially, due to the publication of four fundamental papers:

Fredholm's 1900 paper on Integral Equations

Henri Lebesgue's thesis of 1902 on Integration

Hilbert's paper of 1906 on Spectral Theory

Maurice Frechét's thesis of 1906 on Metric Spaces.

HENRI LEBESGUE 1875-1941

Reformed integration theory so that limits and convergence *of functions* could be handled efficiently. The Riemann integral (of which the Lebesgue integral is an extension) behaves very poorly under limit-taking.

Measure theory and all of Modern Analysis come from this, and the study of measures spawned numerous fundamental questions in Logic, Set Theory, Topology, Geometry.

MAURICE FRECHÉT 1878-1973

In his 1907 paper he co-discovered (independently) with Riesz that the dual of a Hilbert space of functions can be identified with itself.

He began (1906 and after) the general study of metric spaces and metric topology.

Frigyes Riesz 1880-1956 and Johann Radon 1887-1956

Riesz Representation Theorem (1907 and 1909)

Riesz-Fischer theorem (1907)

Radon-Nikodym theorem 1913 (domain space $\mathbb{R}^n$)

Frigyes Riesz

Johann Radon