Hilbert Space Methods Used in a First Course in Quantum Mechanics A Recap

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Why Are We Here?

- We will examine a listing of main theorems and definitions of those purely mathematical results necessary to phrase and understand elementary quantum mechanics in the modern style.
- Frequently, there will be some feature of a proof itself which is useful to emphasize here and in that case we do give a proof, but otherwise reference to a proof will suffice.
- We will use sufficient generality so that the vector spaces used in the applications we have in mind can be seen in their natural mathematical context.
- We will strive to be very clear about what the theorems actually state: what the assumptions are, what the conclusions are ... exactly.

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Quote From Wikipedia

Along the way, I personally would really really like to understand the following snippet, how it is implemented, how mathematical properties are to be interpreted physically, taken from a Wikipedia article on C^*-algebra:

> *In quantum mechanics, one typically describes a physical system using a C^*-algebra $\mathbf{A}$ with unit element; the self-adjoint elements of $\mathbf{A}$ (elements $\mathbf{x}$ with $\mathbf{x}^* = \mathbf{x}$) are thought of as the observables, the measurable quantities, of the system.*
>
> *A state of the system is defined as a positive functional ϕ defined on $\mathbf{A}$ (that is, a complex-linear functional ϕ for which $\phi(\mathbf{x}\mathbf{x}^*) \geq 0$ for all $\mathbf{x}$ in $\mathbf{A}$) such that $\phi(1) = 1$.*
>
> *The expected value of the observable $\mathbf{x}$, if the system is in state ϕ, is then $\phi(x)$.*

Quote From Landsman

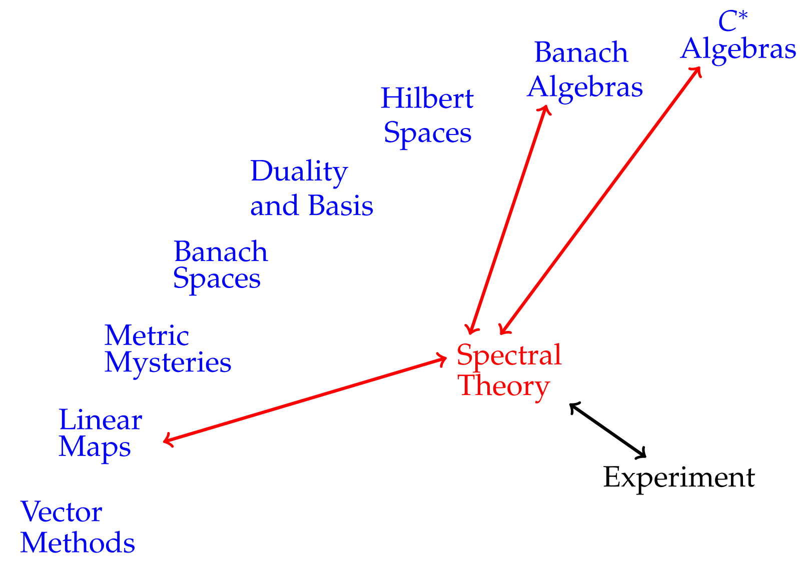
Here is another, this time from some draft notes

Lecture Notes on Hilbert Spaces and Quantum Mechanics

by N. P. Landsman:

In Quantum Mechanics a projection p is a so-called ***yes-no*** *question ... if* $\langle f, f \rangle = 1$*, the answer is yes with probability* $\langle f, pf \rangle$ *and no with probability* $\langle f, (I-p)f \rangle = 1 - \langle f, pf \rangle$.

The Arc of Quantum Wisdom



The Main Vector Spaces

- Function Spaces $\mathbb{F}^B$ with pointwise operations.
- Function Spaces V^B with pointwise operations, where V is a previously defined vector space.
- Sequence Space $\mathbb{F}^{\mathbb{N}}$ with subspaces ℓ_1 and ℓ_2 and ℓ_∞ and c_0 and c_{00}.
- Function Space $\mathbb{F}^X$ where X is a topological space such as $[0,1]$ or $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{R}^n$. These have subspaces such as
  $C(X)$: the continuous $\mathbb{F}$-valued functions on X,
  $M(X)$: the measurable functions,
  $K(X)$: the continuous functions with compact support,
- Quotient Spaces V/W where W is a vector subspace. Elements are sets of members of V which differ by a member of W. A member of V/W can be represented by $v + W = \{ v + w \mid w \in W \}$, but usually it is just given by v alone, with the $+W$ part understood.

Types of Convergence 1

Convergence ideas were introduced when Butch talked about metrics. All our notions of convergence will come from either one metric or involve a pre-determined set of pseudometrics. Usually these pseudometrics come from seminorms.

- The supremum norm is defined on any member f of $\mathbb{F}^B$ by $\|f\|_{sup} = \sup\{ |f(x)| \mid x \in B \}$. A sequence g_n converges to f in this norm if $\| f - g_n \|_{sup} \to 0$.
- The point seminorm at $x \in B$ is defined on any member f of $\mathbb{F}^B$ by $\| f \|_x = | f(x) |$. A sequence g_n converges to f pointwise if $\| f - g_n \|_x \to 0$ for every $x \in B$, one x at a time. This is a weaker notion of convergence.
- The p-norm, where $1 \le p < \infty$, is defined on $\mathbb{F}^{\mathbb{N}}$ by

$$\| s \|_p = \left(\sum_{i \in \mathbb{N}} |s_i|^p \right)^{\frac{1}{p}} .$$

Types of Convergence 2

- The p-seminorm, where $1 \leq p < \infty$, is defined for each positive measure μ on measure space X and $f \in M(X)$ by
$$\|f\|_p = \left(\int_X |f|^p d\,\mu \right)^{\frac{1}{p}}.$$
Usually one identifies the set N_μ of functions whose p-seminorm is 0, the μ-null functions. Then $M(X)$ is replaced by $M(X)/N_\mu$.
So $\|f\|_p = \|g\|_p$ whenever $f - g \in N_\mu$ so the above can be regarded as a *norm*, not a seminorm, on $M(X)/N_\mu$ where you evaluate $\|f + N_\mu\|_p$ by using any representative f of the class and calculating $\|f\|_p$.
In either case a sequence g_n is said to converge to f if
$$\|\, f - g_n \,\|_p \to 0.$$

Topological Vector Spaces I

- Sequence Space ℓ_1 consists of those members of $\mathbb{F}^X$ with finite 1-norm and notion of convergence given by that norm.
- Sequence Space ℓ_2 consists of those members of $\mathbb{F}^X$ with finite 2-norm and notion of convergence given by that norm.
- Sequence Space ℓ_∞ consists of those members of $\mathbb{F}^X$ with finite supremum norm and notion of convergence given by that norm.
- Sequence Space c_{00} is a subspace of all of these, while sequence space c_0 contains ℓ_1 and ℓ_2 and is contained in ℓ_∞ so norms vary (and must be specified) to create interesting examples (and counterexamples.)

Topological Vector Spaces II

The continuous $\mathbb{F}$-valued functions on X, denoted $C(X)$, can have various useful norms and families of seminorms with different properties.

- First of all, we could simply give $C(X)$ the supremum norm.
- But other norms are useful. Let $\mathcal{K}$ denote the set of all (nonempty) compact subsets of X. Define, for each compact set $S \subset X$, the seminorm $\|\cdot\|_S$ to be supremum norm on a function in $C(X)$ restricted to S. We can say that a sequence g_n converges to f if $\|\, g_n - f \,\|_S \to 0$ for every $S \in \mathcal{K}$.
- We can also just use the one-point compact sets: we can say that a sequence g_n converges to f if $\|\, g - f \,\|_x \to 0$ for every $x \in X$, the topology of pointwise convergence.

These topologies yield *different* concepts of convergence.

Topological Vector Spaces III

- If we have a measure μ on X we can also use the p-seminorm, and say that g_n converges to f if $\|\, g_n - f \,\|_p \to 0$. But seminorms are awkward: distinct vectors that cannot be distinguished by the distance-determination-tool lead to endless repetition of phrasing gymnastics.
- So for each p and measure μ we can define the subspace $\mathcal{L}_p(\mu)$ of the quotient space $M(X)/N_\mu$ to consist of those (equivalence classes) of functions f for which $\|\, f \,\|_p < \infty$. Then $\|\cdot\|_p$ becomes a norm, not a seminorm.
- We can also define the essential supremum norm on $M(X)/N_\mu$ by
$$\|f\|_{\text{ess-sup}} = \sup\{\, t \mid \mu(\, |f|^{-1}(\,(t,\infty)\,)\,) > 0 \,\}.$$
$\mathcal{L}_\infty(\mu)$ consists of those (equivalence classes) of functions f for which $\|f\|_{\text{ess-sup}} < \infty$ with this norm.

Topological Vector Spaces IV

- In each case above we defined a specific set of vectors and then a specific family of seminorms to determine convergence.
- Sometimes there was just one seminorm (a small family.) And if that seminorm was not a norm we take a quotient by the vector subspace of those vectors indistinguishable from 0 and then we have a normed space.
- Sometimes the family was large, such as the collection of pointwise seminorms on $C(X)$.
- Whenever a vector space has a topology (i.e. a notion of convergence) determined by a family of seminorms, and if that family allows you to distinguish any two unequal vectors, it is called a Locally Convex Space.

  All of the vector spaces we have discussed are (or were turned into) Locally Convex Spaces.

Banach and Hilbert Spaces

- If V is a Locally Convex Space determined by a countable family of seminorms a sequence s in V is called Cauchy if it is Cauchy with respect to every one of these seminorms. That means for each seminorm $\| \cdot \|_\alpha$ (just one α at a time) if $\varepsilon > 0$ there is an integer N so that whenever $m, n > N$ we have $\| s_n - s_m \|_\alpha < \varepsilon$.
- We say V is complete if every Cauchy sequence converges to a point of V.
- If the family of seminorms consists of a single norm and if V is complete then we call V a Banach Space.
- If, further, that norm satisfies the parallelogram law then it comes from a unique inner product and we call V a Hilbert Space.

The Dual and Operator Norm

- If V is a Banach Space the set of continuous linear functionals on that space will be denoted V'. If W is another Banach Space the continuous linear maps from V to W will be denoted $\mathcal{L}(V, W)$.
- $\mathcal{L}(V, W)$ has a norm, called operator norm, and is itself a Banach Space with that norm.
- This includes V' as a special case: the field of the vector space is itself a (one dimensional) Banach space.
- If V is a Hilbert Space we saw that V' with this norm is too, and there is a conjugate linear isometry identifying V with V' through

$$v \in V \quad \longleftrightarrow \quad \langle \cdot , v \rangle.$$

Convergence of Sequences of Operators

- If V is Hilbert and T_n is a sequence of members of $\mathcal{L}(V, V)$ we say that T_n converges to a continuous linear map S uniformly, or in operator norm, if $\| S - T_n \|$ converges to 0.
- We say that T_n converges to S strongly, or in the strong operator topology, if $\| S(v) - T_n(v) \|$ converges to 0 for each $v \in V$.
- We say that T_n converges to S weakly, or in the weak operator topology, if $| \phi(S(v) - T_n(v)) |$ converges to 0 for each $v \in V$ and each $\phi \in V'$.

  Because of the opportunity to represent functionals using the inner product in a Hilbert Space, this is equivalent to

$$\langle S(v) - T_n(v) , w \rangle \to 0 \quad \text{for all } v, w \in V.$$

Convergence of Sequences of Operators 2

All three of these modes of convergence make $\mathcal{L}(V, V)$ into a Locally Convex Space: or rather three different Locally Convex Spaces.

uniform	strong	weak
$\| S - T_n \| \to 0$	$\| S(v) - T_n(v) \| \to 0$	$\| \phi(S(v) - T_n(v)) \| \to 0$
one norm	one norm for each v	one norm for each v and ϕ.

These modes of convergence are progressively weaker, left to right. Uniform convergence is ideal. Strong convergence is often all you can get, and sufficient for the application. Weak convergence plus other conditions can often be promoted to one of the stronger forms of convergence.

Systems of Equations

- When solving a system of equations of the form

$$\frac{d}{dt}x(t) = A\,x(t) \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}^n$$

  we generate solution

$$x(t) = e^{At}x_0.$$

- This is one of those true-but-not-very-useful solutions: it is hard to calculate e^{At} unless A is diagonal, or near to it.
- But if A is self-adjoint then there is an orthogonal matrix U with $D = U^tAU$ diagonal. This is the spectral theorem. So the series

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \cdots$$

  can be calculated.

Systems of Equations 2

- The convergence of

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \cdots$$

  requires that the entries of A be bounded, and the resulting gigantic sums must converge, but it is easy to show they do **because the matrix is finite.**
- In fact any function represented by a power series can be evaluated at A so long as the appropriate norm of A is less than the radius of convergence of the series.

The Schrödinger Equation

- To solve $i\,\frac{d}{dt}\psi(t) = H\psi(t)$ with $\psi(0) = \psi_0$ we need to calculate

$$e^{-itH} = I - itH + \frac{(itH)^2}{2} - \frac{(itH)^3}{6} + \cdots$$

  but here we have many new issues.
- What does it mean for a series to converge when H is an operator on an infinite dimensional space? What if H is unbounded? Can one make any sense at all out of such a sum in that case? Why should even a single entry of an infinite-matrix analogue be meaningful? And how does the self-adjointness condition come into all this? And the spectral theorem?
- Stay tuned for next quarter.