OF SETS AND STRINGS AND SEALING WAX
AN EPISTLE TO THE PHYSICAL SCIENTISTS AMONG US

LARRY SUSANKA

There are several points that those of us with pure mathematics orientation have neglected to bring up, and it wouldn't hurt to say a word now. Better late than never. It is regarding mathematical S.O.P. and a clash of cultural imperatives.

Mathematical objects are abstract objects, creatures that live only in the dreams of mathematicians. Unlike physicists, we do not have nature to give substance to our mental meanderings, and nature cannot certify consistency for us, as it seems to do for those working in the physical sciences.

Physicists *know* they are talking about something. They can see it, feel it, measure it. They presume nature is consistent, and if some mathematical model fails in that way it is the fault of the model, not their subject. The methods they use *must* be consistent to the extent they match reality. And if a model does not match reality it's no good anyway, so who cares what it does? Physics problems are quite hard enough, thank you very much. Why go looking for trouble?

But mathematicians have been burned, individually and collectively, badly, by unsound foundations and we are burdened with a certain amount of insecurity: we really don't want to spend a year of our lives talking about *nothing*, and that is a distinct possibility with our ethereal objects-of-the-mind.

So we have habits adopted through training and experience that will minimize this risk, and we rely on them; these habits become reflex. Even *with* these habits and ways of speaking we *still* make blunders. That is why we are so . . . , well I guess the word *horrified* applies, when we see folks slinging around these abstract critters without a care in the world.

We have ways, ordinary human ways, sometimes several complementary ways, of thinking about our creations. But we also have structure.

Blended meaning and glorious ambiguity, so important in many artistic endeavors, may drive our inspiration but *cannot be honored in our structure.*

A mathematician's house is built from sets, starting believe-it-or-not with the empty set, and the rules of set formation have rigid grammar that make it as hard as possible to form *grammatically correct* houses that cannot stand, and to *expose conundrums* as rapidly as possible.

Date: February 3, 2013.

The ideal, of course, would be to create a language with which we could talk about anything of interest, in principle prove or disprove any unambiguous conjecture that beckons us, and in which it is provably impossible to form a grammatically correct but inconsistent argument. Alas, that cannot be, for several reasons having to do with the work of a logician named Kurt Gödel.

But we do the best we can.

Here is how mathematicians work:

We see related patterns that interest us. We think for a long time about the key features of these patterns. When we imagine we are ready, we posit a set that is to "carry" the pattern. Then we define an explicit structure which is usually a function of some kind involving this set. Then we show how the patterns that inspired this work are all manifested in properties of this structure. Then we prove theorems about the structure which, of course, apply to the patterns that spawned the whole operation. We then look for new things in the structure that never occurred to us in the original settings, and analogies with other structures we think we already understand.

The "payoff" is two-fold. First, we clarify exactly what we meant when we thought those original patterns were related. This is "librarianship" and is useful to keep things compact and organized. Second, we have the "new stuff" we can find when we look this way, new properties and relationships and analogies no one has thought of before, places where the pattern might be hiding in plain sight.

So here we are, and the latest example is metrics. This is a particularly useful concept, and Butch led us down this road to limits and continuity and simple but somewhat surprising examples.

The question remains: Why would anyone care about metrics? It occurs to me that we didn't ever use the phrase "optimization" and how the word "close" can mean different things depending on what you are trying to do. Here are a few more examples with some quasi-real-world context.

1. Google Search Metric. Our set is "$100,000$ dimensional space" $V = \mathbb{N}^{100,000}$.

Each dimension here represents a different english word. Each member of V represents word count from an article or book at each word. So any item in a library can be assigned to a member of V. The $100,000$-tuple

$$(2, 0, 0, 44, \dots, 1)$$

represents an article that has two instances of the word "aardvark" and a single instance of the word "zymurgy." Two items which have exactly the same word count at every word can't be distinguished, though in principle they could be about entirely different topics. In practice this would be pretty unlikely.

The metric we have in mind is the angle between two members of V defined by dot product. This one would be a pseudometric, and V is bounded (by π) with this metric. Is V compact?

2. $V = \{$ all property parcels inside Seattle city limits $\}$.

Three metrics involving getting around on the streets of Seattle in your car (while obeying posted speed laws and assuming for simplicity certain symmetries that might or might not hold in life.)

The distance between two parcels is the ...

Metric 1: minimum time it takes to get from one parcel to the other.

Metric 2. minimum distance needed to get from one parcel to the other.

Metric 3. least volume of gas required to get from one parcel to the other

Are these metrics bounded? Is V compact with these metrics?

3. In $\mathbb{R}^2$ let the first coordinate stand for pressure and the second stand for temperature. A curvy path segment through space represents information about the possible states of a machine we want to understand. Pick a point not on the curve. You can't actually achieve those values with this machine, but you can get close. Butch gave different metrics in the plane in his talk, and each could give a different answer to "how close can we get to the target state?" There is absolutely no reason in pressure-temperature space that the Euclidean distance would be the "right" notion of distance.

This will also lead to an example of why we care about continuity or compactness: if the curve does not have extra qualities involving these, the machine might not be able to actually achieve a minimum distance at all!

4. Programmer managers are always trying to find metrics to measure efficiency of their workforce. In this case V could be the set of programs written by the workforce. Managers talk about "metrics" to gauge the quality of programs. Should bonuses be awarded based on "number of lines of code?" Probably not. But how do you consistently weigh differences in output quality then? Define a metric!

So the take-away idea here is that mathematicians, peculiar timid creatures that they are, need a dense thicket of sets and functions to feel safe.

This is the journey we are on, and if you, our physical-scientist brothers and sisters, hang on I really do think it will be fun and worth it.

And you are of course critically needed when we get to the upper part of the Arc of Quantum Wisdom.