The Remaining Über-Theorems of Functional Analysis and Remarks About Closed, Self-Adjoint and Essentially Self-Adjoint Operators

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1. The Über-Theorems

Results of this section are direct consequences of the Baire Category Theorem.

1.1. ***Proposition***. ***The Open Mapping Theorem***
If $\Psi \in \mathcal{L}_{\mathbb{F}}\left(V^{Banach}, W^{Banach}\right)$ *and* $\Psi(V) = W$ *then* Ψ *is an open map.*

1.2. ***Corollary***. *Any bounded one-to-one linear map* $\Psi\colon V^{Banach} \to W^{Banach}$ *onto* W *has a continuous linear inverse function.*

1.3. ***Proposition***. ***The Banach-Steinhaus Theorem****, also called* ***The Principle of Uniform Boundedness***

Suppose $\mathcal{A} \subset \mathcal{L}_{\mathbb{F}}\left(V^{Banach}, W^{NLS}\right)$.

Then one of two alternatives apply.

Either $\exists M < \infty$ *for which* $\|\Psi\| \leq M \ \forall \Psi \in \mathcal{A}$

or *the function* $\lambda\colon V \to [0, \infty]$ *given by* $\lambda(x) = \sup_{\Psi \in \mathcal{A}} \|\Psi x\|$ *is infinite on a dense* G_δ *subset of* V.

Recall that a relation is defined to be a set of ordered pairs. The domain of the relation is the set of all first components of any pair in the relation. And a function is a relation for which there is exactly one pair in the set with any domain member as first component.

If $f\colon X \to Y$ where X and Y are topological spaces, the **graph of f** is the set $\{\, (x, f(x)) \mid x \in X \,\} \subset X \times Y$ with subspace topology from the product topological space. The graph of f will be denoted $\gamma(\mathbf{f})$.

So the difference between a function f and its graph $\gamma(f)$ is simply the additional structure of a topological space possessed by the latter. As *sets* they are the same and the properties of $\gamma(f)$ as a topological space can usually be phrased in terms of properties of X and Y, and we will take that approach in the following proposition.

If Y is T_2 and f is continuous, then f must be closed as a subset of $X \times Y$, essentially by definition of continuity. Sometimes the converse implication holds.

1.4. ***Proposition***. ***The Closed Graph Theorem***

Suppose $\Psi \in \mathcal{H}_{\mathbb{F}}(V^{Banach}, W^{Banach})$.

$\Psi \in \mathcal{L}_{\mathbb{F}}(V, W)$ *if and only if*
Ψ *is a closed subset of* $V \times W$ *with product topology.*

2. Closed Operators

Some of the main applications of functional analysis will involve linear functions defined on a subspace, and not all of, a particular normed space. Many of these *cannot* be extended to a continuous function defined on the whole ambient space but have other properties that will, nonetheless, give us some traction.

To that end we consider linear $T\colon \Delta_T \to W$ where $\Delta_T = Domain(T)$ is a subspace of V and both V and W are normed spaces. So Δ_T is a normed space with restriction norm, though it may not be closed in V.

Recall the nature of a function: in our case, function T *is* a set of ordered pairs of the form $(\,x, T(x)\,) \in V \times W$ where x is restricted to come from Δ_T. We will not distinguish here between a function and its graph.

T is called a **closed linear operator** if T, thought of as this set of ordered pairs, is a closed subset of $V \times W$ where $V \times W$ is given product norm. That means that whenever $(x_n, y_n) \in T$ for all $n \in \mathbb{N}$, and if this sequence converges to (a, b) in $V \times W$, then $a \in \Delta_T$ and $b = T(a)$.

If T is closed, this does *not* imply that Δ_T is closed.

It implies that if each x_n is in Δ_T and if this sequence converges to a limit $a \in V$ then a must be in Δ_T *provided that the sequence* $T(x_n)$ *converges to a limit in* W. Only in that case does the requirement that a belong to Δ_T and $b = T(a)$ apply.

This definition does not involve sequences x_n for which $T(x_n)$ is not Cauchy, or when $T(x_n)$ has no limit because W fails to be complete, whether or not x_n converges. If there are two sequences x_n and y_n which both converge to a and $T(x_n)$ converges to b but $T(y_n)$ fails to converge it is only the limit of convergent $T(x_n)$ that is involved in this definition.

2.1. ***Corollary***. *Suppose* $T\colon \Delta_T \to W$ *is linear where* Δ_T *is a subspace of* V*, and both* V *and* W *are normed spaces.*

(i) If T *is closed and* Δ_T *and* W *are Banach then* T *is continuous.*

(ii) If T *is closed and continuous and* W *is Banach then* Δ_T *is closed in* V*.*

(iii) If T *is continuous and* Δ_T *is closed then* T *is closed.*

(iv) Continuity of T *alone does not imply* T *is closed.*

Proof. (i) restates (half of) Proposition 1.4. (ii) and (iii) are left to the reader. A counterexample showing (iv) is given by the identity map restricted to a dense subspace of infinite dimensional V. (Can you produce such a subspace? The finite linear combinations of members of any Schauder basis will do it, but we don't investigate these until a later section.) □

2.2. ***Exercise***. *Suppose* $T\colon \Delta_T \to W$ *is one-to-one and closed where* Δ_T *is a subspace of* V*, and both* V *and* W *are normed spaces. Then the inverse function* $T^{-1}\colon Range(T) \to V$ *is closed.*

2.3. ***Corollary***. *Suppose $T\colon \Delta_T \to W$ is linear where Δ_T is a subspace of V, and both V and W are normed spaces.*

T can be extended to a closed operator whose graph is $\overline{T}$ if and only if $(0, y) \in \overline{T}$ implies $y = 0$.

In terms of T and the original domain, the second condition is equivalent to: For every sequence x_n converging to 0 in Δ_T, either $T(x_n)$ converges to 0 or $T(x_n)$ fails to converge at all.

Proof. The necessity of the second condition is obvious.

So suppose $(0, y) \in \overline{T}$ implies $y = 0$. Since the norm closure of a subspace is itself a subspace, this condition implies that if (a, b) and (a, d) are in $\overline{T}$ then $b = d$: that is, $\overline{T}$ is the graph of some function, which must then be an extension of T. Since $\overline{T}$ is a subspace that function is linear. □

An operator T for which $\overline{T}$ is a function, $\overline{T}\colon S \to W$, is called **closable**. $\overline{T}$ is called the **closure** of T. We emphasize that $S = \Delta_{\overline{T}}$ need not be closed in V.

3. An Example of a Closed Operator

Here we provide an interesting example of a closed operator, the derivative operator. Operators of this general type are, arguably, among the most important closed operators from the applications.

Let $C[a, b]$ denote the Banach space of continuous functions on a closed interval $[a, b]$. For any function, continuous or not, we define supremum norm $\| \cdot \|$ by $\| f \| = \sup\{ f(x) \mid x \in [a, b] \}$. We give $C[a, b]$ this norm. Convergence in this norm is called uniform convergence.

Let $\frac{d}{dx}$ be the differentiation operator defined here, specifically, on the subspace $C^1[a, b]$ of $C[0, 1]$, consisting of continuously differentiable functions on $[a, b]$, where we use one-sided limits to calculate derivatives at the endpoints.

The rest of this section is devoted to showing that if a sequence of continuously differentiable functions converges to a continuous function f, and if the derivatives also converge to a continuous function g, then f is continuously differentiable and $\mathbf{f}' = \mathbf{g}$. The conclusion is then that the derivative operator is closed with this domain.

Suppose (f_n) is a sequence of bounded functions on interval $[a, b]$ and suppose (f_n) is Cauchy in supremum norm. This means that

$$\forall \varepsilon > 0 \text{ there is an integer } N \text{ so that } m, n \geq N \Rightarrow \|f_n - f_m\| < \varepsilon.$$

If (f_n) is Cauchy in supremum norm then $(f_n(x))$ is a Cauchy sequences of numbers for each x, so there is a functions f with

$$f_n \underset{\text{sup}}{\longrightarrow} f$$

where the indicated "sup" convergence means uniform convergence.

We will review several proofs (these are special cases of more general facts proved elsewhere in the appendices) involving continuity, uniform continuity and equicontinuity.

Continuity on a compact interval implies uniform continuity: that is, if $v\colon [a,b] \to \mathbb{R}$ is continuous, then for each $\varepsilon > 0$ there is a $\delta > 0$ so that $x, y \in [a,b]$ and $|x-y| < \delta$ implies $|v(x) - v(y)| < \varepsilon$.

Proof: For each $x \in [a,b]$ find $\delta_x > 0$ so that $y \in [a,b]$ and $|x-y| < \delta_x$ implies $|v(x) - v(y)| < \varepsilon/2$. The set of intervals of the form $[x - \delta_x/2, x + \delta_x/2]$ covers $[a,b]$. Extract a finite subcover $[x_i - \delta_{x_i}/2, x_i + \delta_{x_i}/2]$ for $i = 1, \ldots, k$. Let δ_n be the least of the $\delta_{x_i}/2$ and suppose $|x-y| < \delta_n$.The number y is in one of the $[x_i - \delta_{x_i}/2, x_i + \delta_{x_i}/2]$ so both x and y are in $[x_i - \delta_{x_i}, x_i + \delta_{x_i}]$.

$$\begin{aligned}\text{Then}\quad |v(x) - v(y)| &\le |v(x) - v(x_i)| + |v(x_i) - v(y)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Next we show that the limit f of a sequence (f_n) of continuous functions defined on a compact interval is continuous. This means that $C[a,b]$ is a Banach space with supremum norm.

Proof: Choose N so large that $\|f - f_N\| < \varepsilon/3$. Choose δ so small that $x, y \in [a,b]$ and $|x-y| < \delta$ implies $|f_N(x) - f_N(y)| < \varepsilon/3$.

Now we have the necessary inequality: for $x, y \in [a,b]$ and $|x-y| < \delta$

$$\begin{aligned}|f(x) - f(y)| &\le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

We also need to know that convergence of a sequence (f_n) of continuous functions on a compact interval implies that the sequence is equicontinuous: that is, for each $x \in [a,b]$ we can find $\delta_x > 0$ so that $y \in [a,b]$ and $|x-y| < \delta_x$ then $|f_n(x) - f_n(y)| < \varepsilon$ for every n.

In fact, we have more. Each convergent sequence of continuous functions defined on a compact interval is *uniformly* equicontinuous: the same number δ can be chosen for each $x \in [a,b]$ and every n.

Proof: Choose N so large that $n, m \ge N$ implies $\|f_n - f_m\| < \varepsilon/3$. Choose δ^* so small that if $x, y \in [a,b]$ and $|x-y| < \delta^*$ then $|f_N(x) - f_N(y)| < \varepsilon/3$. Now we have for $m \ge N$

$$\begin{aligned}|f_m(x) - f_m(y)| &\le |f_m(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_m(y)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

Each one of the $f_1, \ldots, f_{N-1}$ is uniformly continuous, so there are positive numbers δ_i so that $x, y \in [a,b]$ and $|x-y| < \delta_i$ implies $|f_i(x) - f_i(y)| < \varepsilon$ for each $i = 1, \ldots, N-1$. So now let δ be the least number in $\{\, \delta^*, \delta_1, \ldots, \delta_{N-1} \,\}$ and uniform equicontinuity of (f_n) follows from this.

Suppose (f_n) is a sequence of continuously differentiable functions on interval $[a,b]$ and suppose that (f_n') is Cauchy in sup-norm. Suppose further that $(f_n(c))$ is a Cauchy sequence of numbers for some $c \in [a,b]$, which therefore converges to some number which we denote $f(c)$.

$(f'_n(x))$ is a Cauchy sequence of numbers for each x so there is a function g, which we saw above must be continuous, for which

$$f'_n \underset{\sup}{\longrightarrow} g.$$

We note that, for any continuous $v\colon [a,b] \to \mathbb{R}$, if $\|v\| < \varepsilon$ then basic facts about the Riemann integral imply that for every $x, c \in [a,b]$

$$\left| \int_c^x v(y)\,dy \right| < \varepsilon\,|x-c|.$$

Now consider the situation with our sequences of functions and derivatives. For each n

$$f_n(x) = f_n(c) + \int_c^x f'_n(y)\,dy$$

Since f'_n converges uniformly to continuous g, and $f_n(c)$ converges to the number $f(c)$, the right hand side converges (using the remark of the preceding paragraph) to $f(c) + \int_c^x g(y)\,dy$. That implies the sequence of numbers on the left side converges to a number we denote $f(x)$ for each $x \in [a,b]$.

By the Fundamental Theorem of Calculus, f is differentiable and $f'(x) = g(x)$.

$$\begin{aligned} \text{Also}\quad |f_n(x) - f(x)| &= \left| \int_c^x f'_n(y)\,dy f_n(x) - \int_c^x g(y)\,dy \right| \\ &= \left| \int_c^x (\, f'_n(y) - g(y)\,)\,dy \right| \leq \|f'_n - g\|\,(b-a). \end{aligned}$$

Since the last term converges to 0 we have $\lim_{n\to\infty} \|f_n - f\| = 0$.

So under the indicated circumstances both (f_n) and (f'_n) converge uniformly to continuous functions f and g, respectively. The function f is continuously differentiable, and in fact $f' = g$. Thus

$$f'_n \underset{\sup}{\longrightarrow} f'.$$

We have just shown that the operator $\frac{d}{dx}\colon C^1[a,b] \to C[a,b]$ is closed.

Actually, we have shown more: closure requires only that *in case* $f_n \underset{\sup}{\longrightarrow} f$ for a sequence f_n of continuously differentiable functions and some *continuous function* f and *if we know* the sequence (f'_n) converges *to a continuous function* g, then f must be continuously differentiable and $f' = g$.

We actually showed that for any sequence of continuously differentiable functions, if the derivative sequence is just Cauchy then we concluded that the derivative sequence *must* converge uniformly to a continuous limit g. And if the derivative sequence is Cauchy and the sequence (f_n) converges at just a single domain value then we *concluded* it converges uniformly to a continuous function f. We then showed the pertinent point: f is differentiable and $f' = g$.

The calculations above show that the operator $\frac{d}{dx}$ is closed in $C[a,b] \times C[a,b]$, but of course $C^1[a,b]$, the domain of $\frac{d}{dx}$, is not closed in $C[a,b]$. The Weierstrass Approximation Theorem tells us that $C^1[a,b]$ is a dense subspace of $C[a,b]$ with supremum norm: $\overline{C^1[a,b]} = C[a,b]$.

The sequence of functions given by $\mathbf{f_n(x) = \frac{\sin(nx)}{n}}$ converges uniformly to 0 but $\mathbf{\frac{df_n}{dx} = \sin(nx)}$ does not converge to 0. Why does this not interfere with our conclusion that this operator is closed?

4. Unbounded Operators

Until now, our concentration has been on bounded (i.e. continuous) operators. But many important operators, differential operators among them, are unbounded, defined on a dense subset of a Hilbert space but not on the whole space.

We confine attention to functions defined to a subset of a Hilbert space $\mathcal{H}$ with values in $\mathcal{H}$.

For each such T we will let Δ_T denote the domain of T. Then for two functions S and T we have a partial ordering given by containment. So $S \subset T$ provided that T is an extension of S to larger domain. In particular, $\Delta_S \subset \Delta_T$ and $S(x) = T(x)$ for every $x \in \Delta_S$.

In the following, we will consider functions defined on vector subspaces of a Hilbert space $\mathcal{H}$. We have certain requirements on any function T in this section.

- $T\colon \Delta_T \to \mathcal{H}$ is linear on vector subspace Δ_T of $\mathcal{H}$.
- We require Δ_T to be ***dense*** in Hilbert space $\mathcal{H}$.
- We assume the field to be the complex numbers.
- We specifically ***do not*** require T to be bounded.

Generally, an operator satisfying these conditions is called an **unbounded operator.** It is an awkward phrase, as we do not exclude the possibility that such an operator is bounded, but it may not be. And many of the operators from important applications will not be.

In our context an unbounded operator T is closed when T is a closed subset of $\mathcal{H} \times \mathcal{H}$ with its natural inner product. This means that whenever sequence x_i, $i \in \mathbb{N}$, in $\mathcal{H}$ converges to a point a and provided $T(x_i)$ converges to a point b then $a \in \Delta_T$ and $T(a) = b$. This condition is implied by continuity, but does not require continuity.

T is closed when $T = \overline{T}$, and T is called closable if the relation $\overline{T}$ is a function. We saw in Corollary 2.3 that this will happen exactly when $(0, y) \in \overline{T}$ implies $y = 0$. In terms of Δ_T, this condition means that whenever x_i converges to 0 in Δ_T then either $T(x_i)$ converges to 0 or $T(x_i)$ fails to converge at all.

If $\overline{T}$ is a function then $\Delta_{\overline{T}}$ contains every member a of $\mathcal{H}$ for which there is *any* sequence x_i in Δ_T converging to a with the property that $T(x_i)$ is also convergent. If this limit is b then $\overline{T}(a) = b$.

This does not imply that $\Delta_{\overline{T}} = \mathcal{H}$!

So in principle, $\overline{T}$ itself could be extended to more of $\mathcal{H}$.

Suppose S is any extension of $\overline{T}$, and suppose $(a, b) \in S$. Since Δ_T is dense in $\mathcal{H}$ there is a sequence x_i from Δ_T with x_i converging to a. If $(a, b) \notin \overline{T}$ then it must be that $S(x_i) = T(x_i)$ fails to converge, and in fact there is no sequence y_i

in Δ_T converging to a for which $T(y_i)$ converges to *anything.* So $b = S(a)$ has no connection to the values of T.

Extensions of T beyond $\overline{T}$ can be made "at random" (subject to linearity) but these extensions, even if they might be good for *something*, cannot be said to have anything to do with T. On the other hand, every point of $\overline{T}$ not already in T is connected to the values of T by a continuity condition at that new domain member, and there is only one possible way of doing this.

So $\overline{T}$ is the smallest closed extension of T, and the only one whose values are *all* connected by a continuity condition to the values of T.

We now proceed to a new issue.

For each $y \in \mathcal{H}$ define the function $A_y(\cdot) = \langle\, T(\cdot)\,,\, y\,\rangle \colon \Delta_T \to \mathbb{F}$.

A_y is a linear functional on Δ_T and *if* it is bounded on Δ_T corresponds to inner product against a unique member $w \in \overline{\Delta_T}$: viz.

$$A_y(\cdot) = \langle\, \cdot\,,\, w\,\rangle.$$

Since Δ_T is assumed dense in $\mathcal{H}$, w is the unique member of $\mathcal{H}$ that "works" for this y, and this uniqueness is one reason to require Δ_T to be dense in $\mathcal{H}$.

We define $T^\dagger(y) = w$ whenever A_y is a bounded functional on Δ_T.

So $T^\dagger$ has its own domain, consisting of all those y for which the functional $A_y(\cdot) = \langle\, T(\cdot)\,,\, y\,\rangle$ is bounded.

The function $\mathbf{T}^\dagger \colon \Delta_{T^\dagger} \to \mathcal{H}$ defined by

$$\langle\, x\,,\, T^\dagger(y)\,\rangle = \langle\, T(x)\,,\, y\,\rangle \quad \forall x \in \Delta_T$$

is called **the adjoint** of T.

$T^\dagger$ is a closed operator for any T. To see this, we suppose x_i is a sequence in $\Delta_{T^\dagger}$ and x_i converges to a and $T^\dagger(x_i)$ converges to b. So

$$\langle\, x\,,\, T^\dagger(x_i)\,\rangle = \langle\, T(x)\,,\, x_i\,\rangle \quad \forall x \in \Delta_T.$$

The left side converges to $\langle\, x\,,\, b\,\rangle$ and the right side to $\langle\, T(x)\,,\, a\,\rangle$ for all $x \in \Delta_T$. That means $|\,\langle\, T(x)\,,\, a\,\rangle\,| \le \|b\|\,\|x\|$ for all $x \in \Delta_T$ so $a \in \Delta_{T^\dagger}$. And the uniqueness condition mentioned earlier implies that $T^\dagger(a) = b$.

We call the operator T as above **symmetric** if $T \subset T^\dagger$: in other words, if $T^\dagger$ is an extension of T. Linear T is symmetric exactly when $T^\dagger(x)$ is defined for every $x \in \Delta_T$ and $T(x) = T^\dagger(x)$.

If $T \subset T^\dagger$ then $T^{\dagger\dagger} \subset T^\dagger$, but there is no reason to think that $T^\dagger \subset T^{\dagger\dagger}$. The adjoint of a symmetric operator might fail to be symmetric.

If the original operator had been bounded on Δ_T then it could be extended in a unique way to a continuous operator defined on all of $\mathcal{H}$. In that case, $T^\dagger$ is the ordinary Hilbert space adjoint, and the symmetry condition just means that this unique extension of T to all of $\mathcal{H}$ is self-adjoint. Bounded or not, the domain of $T^\dagger$ for symmetric T will often be larger than the domain of T.

4.1. ***Lemma***. *(i) The adjoint of an unbounded operator is closed.*
(ii) So a symmetric operator is always closable.
(iii) If T is symmetric and $\Delta_T = \mathcal{H}$ then T is bounded.

PROOF. See the preceding remarks and, for (iii), the closed graph theorem. □

Item (iii) of Lemma 4.1 is called **The Hellinger-Toeplitz Theorem.** So if we know that symmetric T is *not* bounded, then we know that the domain of $T^\dagger$ cannot be all of $\mathcal{H}$.

A symmetric T *might* have the same domain as $T^\dagger$.

If $T = T^\dagger$ we say T is **self-adjoint**. It is to these unbounded operators that some of our most important theorems, such as the Spectral Theorem and Stone's Theorem on one-parameter unitary groups, apply.

We say that T is **essentially self-adjoint** if it has exactly one self-adjoint extension.

To be proved later...

- If T is symmetric and there is some non-real λ for which both $(T - \lambda I)(\Delta_T)$ and $\left(T - \overline{\lambda} I\right)(\Delta_T)$ are dense in $\mathcal{H}$ then $\overline{T}$ is self-adjoint, and is the unique self-adjoint extension of T.
- If T is symmetric and there is some non-real λ for which neither λ nor $\overline{\lambda}$ is an eigenvalue for $T^\dagger$ then $\overline{T}$ is self-adjoint, and is the unique self-adjoint extension of T.
- Observables, such as position and momentum operators, are essentially self-adjoint *if their domain space is specified properly.* Incorrect choices correspond top "nonphysical" boundary conditions.