BAYES' THEOREM

Fact 1) $P(A \text{ and } B) = P(B|A)\,P(A)$.

Fact 2) $P(B) = P(B \text{ and } A) + P(B \text{ and not} A) = P(B|A)P(A) + P(B|\text{ not}A)P(\text{not}A)$.

Fact 3) $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$.

Bayes' Theorem: $$P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)P(A) + P(B|\text{ not}A)P(\text{not}A)}$$

Example. Suppose you must give your patient worrisome news about a medical test. Your patient tested positive. This test is for a disease that is present in one tenth of a percent of the general population. This is the type of information that the Center for Disease Control in Atlanta makes available for many diseases.

The packaging materials that came with the test give the characteristics of the test: namely its specificity and its sensitivity. These values must be determined before the test can be released for use and are conditional probabilities.

Specificity $= P(-|no\, D)$ and Sensitivity $= P(+|D)$.

For this test the specificity is 97% and the Sensitivity is 99%.

How worried should your patient be? That is, what is $P(D|+)$?

Answer:

$$\begin{aligned} P(D|+) &= \frac{P(+|D)\,P(D)}{P(+|D)P(D) + P(+|\text{ no}D)P(\text{no}D)} \\ &= \frac{(0.99)\,(0.001)}{(0.99)(0.001) + (0.03)(0.999)} \approx 0.032. \end{aligned}$$

Now suppose that you find out that your patient has a genetic marker. 2% of people with the marker have the disease. Now how worried should your patient be?

$$\begin{aligned} P(D|+) &= \frac{P(+|D)\,P(D)}{P(+|D)P(D) + P(+|\text{ no}D)P(\text{no}D)} \\ &= \frac{(0.99)\,(0.02)}{(0.99)(0.02) + (0.03)(0.98)} \approx 0.402. \end{aligned}$$