PARTIAL FRACTION DECOMPOSITION

LARRY SUSANKA

1. Facts about Polynomials and Notation

We must assemble some tools and notation to prove the existence of the standard partial fraction decomposition, used as an integration technique in most Calculus texts.

If f is a generic polynomial, $\mathbf{Deg(f)}$ will denote the degree of f.

If f is any polynomial let $\mathbf{c(f)}$ be the leading coefficient of f.

f is called **monic** if $c(f) = 1$.

If p and q are two nonzero polynomials let $\mathbf{GCF(p, q)}$ denote the monic polynomial which is the greatest common factor of p and q.

The theorem states that when f and a are polynomials with real coefficients and no common factors and $Deg(f) < Deg(a)$ then the fraction f/a can be written in a particular form which is amenable to methods of integration learned previously.

For convenience, we presume without loss that a is **monic**.

We assume that a is factored into the product of powers of distinct monic terms which have real coefficients and are either linear or irreducible quadratic.

The form guaranteed by the theorem is that f/a can be written as a (possibly lengthy) sum of terms of the form

$$\frac{\mu}{(x-\beta)^n} \text{ or } \frac{\mu x + \gamma}{(x^2 + \delta x + \epsilon)^n} \text{ where } (x-\beta)^n \text{ and } (x^2 + \delta x + \epsilon)^n$$

are factors of a and each $x^2 + \delta x + \epsilon$ is irreducible (that is, it has complex roots) and all the coefficients and constants involved are real.

This theorem doesn't tell you how to find the μ or $\mu x + \gamma$ for each summand, nor does it mention the existence of a (unique) factorization of the denominator a into the product of irreducible powers.

The existence of the factorization follows from the Fundamental Theorem of Algebra, though the proofs are not constructive in the sense that they don't tell you *how* to find these factors in the way that the quadratic formula does for second degree polynomials.

We will describe a general proof technique and two theorems from algebra.

The proof technique is called **Proof by Induction.** If you have a sequence of propositions $P_1, P_2, P_3, \ldots$ indexed by the natural numbers you may conclude that

Date: November 13, 2015.

every proposition on this list is true provided you can show (I) P_1 is true and (II) If P_k is true for all $k < n$ then P_n must be true.

The reason we can draw this conclusion is that if some of the P_j are false there would have to be some smallest integer n for which P_n is false. That n cannot be 1 by (I), and (II) then implies P_n is true. This contradiction implies that the premise that led to it, namely that there are false P_j, must *itself* be false. In other words all P_n are true.

With this in hand, we give the two algebraic results.

First, is the Euclidean algorithm. It states that if p and h are polynomials and $h \neq 0$ there are polynomials q and r with $r = 0$ or $Deg(r) < Deg(h)$ and $p = qh + r$.

Second, if the greatest common factor of two nonzero polynomials g and h is w then there are polynomials c and d for which $w = cg + dh$. In that case, it follows that c and d share no common factor. The special case we will care about is when $w = 1$: that is, g and h share no common factor.

You can either accept these two results, or follow the arguments for them given now.

The first result is obviously true whenever $p = 0$ or if $Deg(p) < Deg(h)$ since we can choose $q = 0$ and $r = p$.

If the first result were ever false, there would be a nonzero pair p and h exhibiting this, and for which $Deg(p) - Deg(h)$ is as small as possible. That difference cannot be negative, by the discussion we just gave. So the difference $n = Deg(p) - Deg(h)$ must be positive or 0. Then

$$w = x^n \frac{c(p)}{c(h)} h - p$$

is 0 or it is a polynomial that has degree less than the degree of p, so $w = 0$ or $Deg(w) - Deg(h) < n$ and the first result applies: that is, there are polynomials a and b with $b = 0$ or $Deg(b) < Deg(h)$ and $w = ah + b$. But then

$$ah + b = w = x^n \frac{c(p)}{c(h)} h - p \quad \text{so} \quad p = \left(x^n \frac{c(p)}{c(h)} - a \right) h + (-b).$$

But this represents p in the form required by the first result and p and h were specifically picked because they were among the pairs of polynomials for which the first result *fails.* Since that is impossible, it must be that the first result *never fails.*

That is, the first result is true for all p, h pairs.

The argument may be recognized as a (very slightly disguised) Proof by Induction, in this case induction on $Deg(p) - Deg(h)$.

The proof of the second algebraic result can also be handled by Induction.

If p and q are nonzero polynomials let $\mathcal{S}(p, q) = \{\, ap + bq \mid a, b \text{ are polynomials} \,\}$.

Every member of this set of polynomials is divisible by $GCF(p, q)$ and the claim of the second result is that $GCF(p, q)$ is actually a member of $\mathcal{S}(p, q)$.

If q is the constant polynomial, the second result is obviously true.

If the second result were to ever fail it would have to fail for some p, q pair where q has *smallest degree* among all the failed pairs.

By the minimality assumption on q we have $Deg(q) \leq Deg(p)$ so there are polynomials c and r with $r = 0$ or $Deg(r) < Deg(q)$ and $p = cq + r$.

r cannot be 0 because in that case p is a multiple of q and the second result is obviously true in that case. It must then be that $GCF(q, r) = GCF(p, q)$ since every shared factor of p and q must be shared by p and r, and conversely.

Since the degree of r is less than that of q the second result is true for the q, r pair and there are polynomials x and y for which $xq + yr = GCF(q, r) = GCF(p, q)$.

But then $xq + y(p - cq) = yp + (x - yc)q = GCF(p, q)$ contrary to choice of the p, q pair.

In other words, there can be no pair for which the second result fails. It must always be true.

2. The Proof Part One

We will suppose that statement $\mathbf{P_n}$ is:

The Partial Fraction Theorem is true when $\mathbf{\frac{f}{a}}$ has the form

$$\textbf{(i)}\ \frac{\zeta}{(\mathbf{x}-\mu)^{\mathbf{n}}} \qquad \textbf{or} \qquad \textbf{(ii)}\ \frac{\zeta\mathbf{x}+\mu}{(\mathbf{x^2}+\delta\mathbf{x}+\epsilon)^{\mathbf{n}}}.$$

P_1 is true: that is, the theorem is true for f/a of the form

$$(i)\ \frac{\zeta}{x-\mu} \qquad \text{or} \qquad (ii)\ \frac{\zeta x+\mu}{x^2+\delta x+\epsilon}$$

because (i) and (ii) are already in the form specified in the theorem.

We now presume that statement P_k is true for all $k < n$ and some n exceeding 1. This is part (II) of the Proof by Induction process.

Let's examine P_n.

In case (i) there is a polynomial q and real constant r with $f = q(x - \mu) + r$.

$$\text{Then} \quad \frac{f}{(x-\mu)^n} = \frac{q(x-\mu)+r}{(x-\mu)^n} = \frac{q}{(x-\mu)^{n-1}} + \frac{r}{(x-\mu)^n}.$$

The last fraction is in the form specified in the theorem, while the previous term can be placed in the appropriate form by inductive hypothesis (II).

In case (ii) there are polynomials q and (at most) first degree r with

$$f = q(x^2 + \delta x + \epsilon) + r.$$

This time we have:

$$\frac{f}{(x^2+\delta x+\epsilon)^n} = \frac{q(x^2+\delta x+\epsilon)+r}{(x^2+\delta x+\epsilon)^n} = \frac{q}{(x^2+\delta x+\epsilon)^{n-1}} + \frac{r}{(x^2+\delta x+\epsilon)^n}.$$

Again, the last fraction is in the form specified in the theorem, while the previous term can be placed in the appropriate form by inductive hypothesis (II).

So we have proved the theorem when a is a power of factors of these two specified types.

3. The Proof Part Two

Let's examine now the situation in the general theorem where nth degree monic polynomial a has a nontrivial factorization $a = gh$ where g and h are monic and have no common factors and $Deg(f) < n$. So g will have all of some of the irreducible factors of a and h will have all of the remaining irreducible factors.

$$Deg(f) < Deg(g) + Deg(h) = n$$

and both $Deg(g)$ and $Deg(h)$ are at least 1. Since the greatest common factor of g and h is 1 there are polynomials c and d for which $1 = cg + dh$. Note that c can share no factors with h, while d can share no factors with g.

Then

$$\frac{f}{a} = \frac{f(cg+dh)}{a} = \frac{fcg}{a} + \frac{fdh}{a} = \frac{fc}{h} + \frac{fd}{g}.$$

It is possible that both $Deg(fc) < Deg(h)$ and $Deg(fd) < Deg(g)$ in which case both fractions on the end of the line above are in the form to which the theorem applies, but each denominator has *strictly fewer* irreducible factor types in the denominator than has a.

On the other hand, it may be that one of $Deg(fc) \geq Deg(h)$ or $Deg(fd) \geq Deg(g)$ holds.

We will assume that $Deg(fc) \geq Deg(h)$ and the other case is handled identically.

If $Deg(fc) \geq Deg(h)$ then $fc = qh + r$ for certain polynomials q and r where $Deg(r) < Deg(h)$. Note that this implies that r can share no factors with h because c shares no factor with h so any shared factor of r and h would have to be shared also with f, contradicting the assumption that f/a is in lowest terms.

We now have

$$\frac{f}{a} = \frac{f(cg+dh)}{a} = \frac{(qh+r)g + dhf}{a} = \frac{rg + (q+df)h}{a}.$$

Note that $q+df$ can share no factor with g, again because f/a is in lowest terms.

The left-hand side of the last numerator has degree strictly less than $Deg(h) + Deg(g)$, and since f also has degree less than $Deg(h) + Deg(g)$ we must have $Deg(\,(q+df)h\,)$ less than $Deg(h) + Deg(g)$ too.

So $Deg(q+df) < Deg(g)$ and (by its definition) $Deg(r) < Deg(h)$.

Now we have

$$\frac{f}{a} = \frac{rg + (q+df)h}{a} = \frac{r}{h} + \frac{q+df}{g}$$

and both of these fractions are in the form to which the theorem applies, but with strictly fewer types of irreducible factors in the denominator than has a.

4. The Proof Part Three

We will suppose this time that statement $\mathbf{P_n}$ is the assertion

The Partial Fraction Theorem is true when a has
n different types of irreducible factors.

P_1 is true: that is the content of The Proof Part One, above. We have in hand, therefore, step (I) of the Proof by Induction process.

We now presume that statement P_k is true for all $k < n$ and some n exceeding 1. This is the setup for step (II) of the Proof by Induction process.

Let's look at P_n.

In The Proof Part Two above we showed that any fraction in the form of the theorem whose denominator has n different types of irreducible factors can be split into the sum of two fractions of the type to which the theorem applies, but with strictly fewer irreducible factor types in their denominators.

But each of these two terms can be placed in the appropriate form guaranteed by the theorem by inductive hypothesis (II). So their sum can be as well.

We have verified the two steps of the Proof by Induction process, and conclude that the theorem is true for any number of irreducible factors.

□